Section 13

1. On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a p fraction of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Given the results of your experiment, how should you estimate p?

(b) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

2. Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops. We are told that

$$\Pr[N = n | K = k] = \frac{1}{k}, \qquad \text{for } n = 1, \ldots, k$$

(a) Find the joint distribution of K and N.

(b) Find the marginal distribution of N.

(c) Find the conditional distribution of K given that $N = 2$.

(d) We are now told that he bought at least 2 but no more than 3 books. Find the conditional mean and variance of K, given this piece of information.

(e) The cost of each book is a random variable with mean 3. What is the expectation of his total expenditure? *Hint:* Condition on events $N = 1, \ldots, N = 4$ and use the total expectation theorem.

3. Let X be a random variable with a geometric distribution, $X \sim Geom(p)$.

(a) Compute the distribution of X conditioned on the event $X > 1$, i.e. $\Pr[X = i | X > 1]$.

(b) How does the conditional distribution above compare to the unconditional distribution of X? Given this, how is $\mathbb{E}(X|X > 1)$ related to $\mathbb{E}(X)$? Write down $\mathbb{E}(X|X > 1)$ in terms of $\mathbb{E}(X)$.

(c) Using the total expectation rule, we can compute $\mathbb{E}(X)$ in terms of $\mathbb{E}(X|X = 1)$ and $\mathbb{E}(X|X > 1)$. Determine $\mathbb{E}(X)$ from the total expectation rule and your expression for $\mathbb{E}(X|X > 1)$ from part (b).

4. Tired of hosting the same game year after year, Monty Hall decided to make some changes to his game. There are still three doors, but now one contains 1000 dollars, one contains 500 dollars, and one contains 0 dollars, with the order of the prizes randomly permuted. The contestant first selects a door. Then she has the choice of paying X dollars for Monty to open, among the two unchosen doors, the one that contains the smaller amount of money. If the contestant paid Monty, she then has the choice of switching to the other unopened door.

(a) Suppose the contestant refuses to pay Monty. In this case, what is the expected value of her prize?

(b) Suppose that the contestant decides to pay, and then Monty opens a door that contains $\$500$. Given this, what is the expected value of her prize if she switches, and what is the expected value of her prize if she sticks with her original door?

(c) Now for a different scenario: Suppose that the contestant pays, and then Monty opens a door that contains $\$0$. Given this, what is the expected value of her prize if she switches, and what is the expected value of her prize if she sticks with her original door?

(d) Now suppose a second contestant, Bob, decides in advance that he will always pay and always switch to the unopened door (no matter what he sees behind the door that Monty opens). What is the overall expected value of his prize, with this strategy?

(e) What is the most money Monty can charge for opening one of the two unchosen doors and still make it on average profitable for the contestant to pay Monty?

5. If raisins are dropped randomly when making bread, then the number of raisins in a bread loaf is distributed approximately according a Poisson distribution. Suppose you have a stack of n bread loaves all of which came from the same bakery. They were bought from bakery A with probability $\frac{1}{2}$ and bakery B with probability $\frac{1}{2}$. You know that the number of raisins in a loaf from bakery A is Poisson with mean λ_1 and that from bakery B is Poisson with mean λ_2. Being bored at breakfast on a Saturday ~~morning~~ afternoon, you take the n loaves of bread and count the number of raisins in them to be $k_1, \ldots, k_n$. Based on this observation, what is the probability of all the loaves being from bakery A?