Section 12

1. A friend tells you about a course called "Laziness in Modern Society" that requires almost no work. You hope to take this course so that you can devote all of your time to CS70. At the first lecture, the professor announces that grades will depend only a midterm and a final. The midterm will consist of three questions, each worth 10 points, and the final will consist of four questions, also each worth 10 points. He will give an A to any student who gets at least 60 of the possible 70 points.

However, speaking with the professor in office hours you hear some very disturbing news. He tells you that to save time he will be grading as follows. For each students midterm, he'll choose a real number randomly from a distribution with mean $\mu = 5$ and variance $\sigma^2 = 1$. He'll mark each of the three questions with that score. To grade the final, he'll again choose a random number from the same distribution, independent of the first number, and he'll mark all four questions with that score.

If you take the class, what will the mean and variance of your total class score be? Can you conclude that you have less than a 5% chance of getting an A? Why?

2. A spider is expecting guests and wants to catch 500 flies for her dinner. Exactly 100 flies pass by her web every hour. Exactly 60 of these flies are quite small and are caught with probability $\frac{1}{6}$ each. Exactly 40 of the flies are big and are caught with probability $\frac{3}{4}$ each. Assume all fly interceptions are mutually independent. We are trying to find an upper bound on the probability that the spider catches at least 500 flies in 10 hours.

(a) The Markov bound states that for a non-negative random variable X with expectation $\mathbb{E}(X) = \mu$, and any $\alpha > 0$,

$$\Pr[X \geq \alpha] \leq \frac{\mathbb{E}(X)}{\alpha}$$

(see Note 15). What would the Markov bound be on the probability that the spider will catch her quota of 500 flies?

(b) What would the Chebyshev bound be on the probability that the spider will catch her quota of 500 flies?

3. On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a p fraction of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Given the results of your experiment, how should you estimate p?

(b) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

4. Suppose a box has 6 brown balls and 4 purple balls. A random sample of size n is selected with replacement. Let X= "number of brown balls selected." Write the distribution of X. Can this be closely approximated with a Poisson distribution? If so, write the approximate distribution. If not, explain why not. If so, but only under certain conditions, explain these conditions and write the approximate distribution.

5. Now suppose the box has 6 brown balls and 4,000 purple balls. As in the previous exercise a random sample of size n is selected with replacement and X= "number of brown balls selected." Write the distribution of X. Can this be closely approximated with a Poisson distribution? If so, write the approximate distribution. If not, explain why not. If so, but only under certain conditions, explain these conditions and write the approximate distribution.

6. Amir is trying to hitchhike along a deserted road. The probability that a car drives by during any given minute is 0.05.

(a) Let A_i be the event that a car drives by in the i^{th} minute. The events $A_1, \ldots, A_{20}$ are mutually independent. What's the probability that no cars will appear in a particular span of 20 minutes?

(b) Now suppose Amir has waited 20 minutes and no cars have come. What's the probability that no cars will arrive during the next 20 minutes?