Section 10

1. A roll of the dice

Consider a single roll of two dice, one red and one blue.

(a) Let R be the value of the red die. What is the distribution of R? What is the expectation of R?

(b) Let M be the maximum of the numbers on the two dice. What is the distribution of M? What is the expectation of M?

(c) How do the distribution and expectation of M compare to that of R?

(d) Are the events $R = 6$ and $M = 6$ disjoint? Are they independent? What about the events $R = 6$ and $M = 1$?

[Hint: Recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.]

2. To pay or not to pay?

Reese Prosser never puts money in a 25-cent parking meter in Hanover. He assumes that there is a probability of 0.05 that he will be caught. Assume each offense that is caught costs him $10. Under his assumptions:

(a) How does the expected cost of parking 10 times without paying the meter compare with the cost of paying the meter each time?

(b) If he parks at the meter 10 times, what is the probability that he will have to pay more than the total amount he could end up saving by not putting the money?

3. Colorful Marketing Language

A candy factory has an endless supply of red, orange and yellow jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each, with each possible combination of colors in the jar being equally likely. (One possible color combination, for example, is a jar of 56 red, 22 orange, and 22 yellow jelly beans.) As a marketing gimmick, the factory claims that the number of possible combinations is so large that there is negligible probability of finding two jars with the same color combination. You are skeptical about this claim and decide to do some calculations to test it.

(a) Find n, the number of different possible color combinations of jelly beans in a single jar.

(b) In terms of n, write down the probability that two jars of jelly beans have different color combinations.

(c) Again in terms of n, write down the probability that m jars of jelly beans all have different color combinations. [NOTE: You do not need to simplify your expression.]

(d) Approximately how many jars of jelly beans would you have to buy until the probability of seeing two jars with the same color combination is at least $\frac{1}{2}$? [NOTE: You should state your answer only as an order of magnitude (i.e., $10, 100, 1000, \ldots$). You may appeal to any result from class in order to derive your estimate; it should not be necessary to perform a detailed calculation.]

4. Say What?

In a binary communication channel the transmitter sends *zero* or *one*, but at the receiver there are three possibilities: a *zero* is received, a *one* is received, and an *undecided* bit is received (which means that the receiver will ask the transmitter to repeat the bit). Define the event $T_1 = \{$ 1 is sent $\}$ and $T_0 = \{$ 0 is sent $\}$ and assume that they are equally probable. At the receiver we have three events: $R_1 = \{$ 1 is received $\}$, $R_0 = \{$ 0 is received $\}$, $R_u = \{$ cannot decide the bit $\}$. We assume that we have the following conditional probabilities: $\Pr[R_0|T_0] = \Pr[R_1|T_1] = 0.9$, $\Pr[R_u|T_0] = \Pr[R_u|T_1] = 0.09$.

(a) Find the probability that a transmitted bit is received as *undecided*.

(b) Find the probability that a bit is received in error (error means sending *one* while receiving *zero* OR sending *zero* while receiving *one*).

(c) Given that we received a *zero*, what is the conditional probability that a *zero* was sent? What is the conditional probability that a *one* was sent?

5. Locked Out

You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

Find the distribution and the expectation of the number of trials you will need to open the front door, under the following alternative assumptions:

(a) after an unsuccessful trial, you mark the corresponding key so that you never try it again, or

(b) at each trial, you are equally likely to choose any key.