Section 7

1. A city of n people must elect its city council. The council has a president, a vice president, a secretary, and k general members (the k general member positions are identical). How many ways are there to choose the city council from among the n residents?

2. A license plate contains 5 characters (order matters). Each character may either be an upper-case letter A-Z or a number 0-9. How many license plates. . .

    (a) contain only letters?

    (b) have exactly three letters and two numbers?

    (c) contain the string ABC?

    (d) have at least two of the same character?

3. Four natural numbers x_1, x_2, x_3, and x_4 are chosen such that $x_i \leq 20$ for all i. How many ways can the numbers be chosen such that $x_1 + x_2 + x_3 + x_4 \geq 20$?

4. Prove the following identities with a combinatorial proof:

    (a)
$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$$

    (b)
$$\sum_{i=1}^{n} \binom{n}{i} \cdot i = n \cdot 2^{n-1}$$

5. We wish to count the number of ways to throw k balls into n bins. We distinguish one final configuration from another only by the content of the bins, not by the order in which the balls land in the bins.

    (a) First, suppose that the k balls and the n bins are *distinguishable*, i.e. each balls is labeled with a different number and each bin is given a distinct name. How many ways can we throw the balls if we want the first and the second bins to have exactly two balls?

    (b) Suppose balls and bins are still distinguishable, but now $k = 2n$ and we would like every bin to have exactly 2 balls. The following solution is proposed:

        We first place the k balls in a line. Then we throw the 1st and 2nd balls into the first bin, the 3rd and 4th balls into the second bin, etc. Thus, the number of ways to throw the balls is just the number of ways to arrange k distinct balls in a line, which is $k!$.

    Explain why this 'solution' is wrong and give the correct answer.

    (c) We can generalize the previous problem. Suppose we have $r_1 + r_2 + ...r_n = k$ where r_i are all natural numbers. If we want the i-th bin to have r_i balls, how many ways are there to throw the balls?