Section 3

1. Use induction to prove that the following equalities are true for all positive integers n:

(a) $\sum_{i=1}^{n}(4i-3) = n(2n-1)$

(b) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(d) $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

(e) $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

2. Prove that for every positive integer n greater than 2, $3^n > n^2$.

3. Use strong induction to prove that a class of $n \geq 12$ students can be broken into groups where each group has exactly 4 or 5 members.

4. Let m be an even natural number. Find natural numbers x and y such that $m = (x+y)^2 + 3x + y$. Example: for $m = 6$, $x = 0$ and $y = 2$ satisfy the equation. Try a few cases to find a pattern and then use induction to prove that the pattern works.

Hint: Let $m = 2n$, where $n \in \mathbb{N}$. Use induction to find x_n and y_n such that $2n = (x_n + y_n)^2 + 3x_n + y_n$.

5. Use induction to show that for any natural number $n \geq 1$, given pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ of integer numbers, there exist integer numbers c and d such that

$(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots(a_n^2 + b_n^2) = c^2 + d^2$.

Hint: $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.

6. Suppose n is a positive natural number whose prime factorization is: $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 \ldots p_k$ are natural, distinct prime numbers, and $a_1, \ldots a_n$ are positive natural numbers. Use induction to show that the number of divisors of n is $(1 + a_1)(1 + a_2) \cdots (1 + a_k)$. Example: $n = 12 = 2^2 \times 3$ has exactly $3 \times 2 = 6$ divisors.