Section 2

1. Negation and DeMorgan's Law

(a) Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as DeMorgan's Law.

(b) Use a truth table to show that the negation of $P \Rightarrow Q$ is $P \wedge \neg Q$, in another words, $\neg(P \Rightarrow Q)$ is logically equivalent to $P \wedge \neg Q$. What is the negation of $P \Leftrightarrow Q$?

(c) Consider the false statement "For each x in $\mathbb{R}$. $x^2 \geq x$" (consider $0 < x < 1$). What is the negation of this statement? Is it "For each x in $\mathbb{R}$. $x^2 < x$"? Why not? Let $P(x)$ be the proposition "$x^2 \geq x$" with x taken from the universe of real numbers $\mathbb{R}$. Then our original statement is succinctly written as $\forall x.P(x)$. How do we negate this with DeMorgan's Law?

2. Suppose we're considering the domain of just 2 numbers $S = \{0, 1\}$. Try to re-state the following propositions without using any quantifiers. For example, $\forall x.P(x)$ can be re-formulated as $P(0) \wedge P(1)$.

(a) $\exists x.P(x)$

(b) $\neg\exists x.P(x)$

(c) $\forall x.\exists y.P(x, y)$

(d) $\exists x.P(x) \vee (\forall y.Q(x, y))$

(e) $\neg(\forall x.\exists y.P(x) \Rightarrow Q(y))$

3. Rewrite the following statements in propositional logic. (Use $\mathbb{N}$ to denote the set of natural numbers and $\mathbb{Z}^+$ to denote the set of positive integers.)

(a) For all natural numbers n, n is odd if n^2 is odd.

(b) For all natural numbers n, $n^2 - n + 3$ is odd.

(c) There are no positive integer solutions to the equation $x^2 - y^2 = 10$.

4. Let $x_0 = 1$ and $x_1, x_2, x_3 > 0$. Prove by contrapositive that, if $x_3 > 8$, then $\exists\, i \in \{0, 1, 2\}, \dfrac{x_{i+1}}{x_i} > 2$.

5. Prove that $\forall x \in \mathbb{N}$, x is divisible by 3 if and only if the sum of the digits of x is divisible by 3.

6. Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

(I) No one, who is going to a party, ever fails to brush his or her hair.

(II) No one looks fascinating, if he or she is untidy.

(III) Opium-eaters have no self-command.

(IV) Everyone who has brushed his or her hair looks fascinating.

(V) No one wears kid gloves, unless he or she is going to a party.

(VI) A person is always untidy if he or she has no self-command.

(a) Write each of the above six sentences as a quantified proposition over the universe of all people. You should use the following symbols for the various elementary propositions: $P(x)$ for "x goes to a party", $B(x)$ for "x has brushed his or her hair", $F(x)$ for "x looks fascinating", $U(x)$ for "x is untidy", $O(x)$ for "x is an opium-eater", $N(x)$ for "x has no self-command", and $K(x)$ for "x wears kid gloves".

(b) Now rewrite each proposition equivalently using the *contrapositive*.

(c) You now have twelve propositions in total. What can you conclude from them about a person who wears kid gloves? Explain clearly the implications you used to arrive at your conclusion.