CS 70   Discrete Mathematics and Probability Theory
Summer 2011 Kamil   HW 5

Due Tuesday, July 19, 4:59pm

1. (9 pts.)   Counting telephone numbers
For the purposes of this problem, a phone number is an arbitrary sequence of 7 decimal digits.

   (a) A *single-digit number* is a 7-digit phone number made up out of exactly one number. (For instance, 888-8888, 000-0000, and 555-5555 are single-digit numbers.) How many different single-digit numbers are there?

   (b) A *non-repetitious number* is a 7-digit phone number where no digit is used more than once. (For instance, 571-2834, 102-9543, and 019-6273 are non-repetitious numbers, but 523-3678 is not.) How many different non-repetitious numbers are there?

   (c) A *taxicab number* is a 7-digit phone number made up out of exactly two different digits. (For instance, 888-5858, 626-6666, 525-5252, 511-5115, and 000-1001 are taxicab numbers. 718-7818 and 777-7777 are not taxicab numbers.) How many different taxicab numbers are there?

2. (8 pts.)   Algebraic vs. combinatorial proofs
Consider the following identity:
$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

   (a) Prove the identity by algebraic manipulation (using the formula for the binomial coefficients).

   (b) Prove the identity using a combinatorial argument.

3. (14 pts.)   Sample Space and Events
Consider the sample space Ω of all outcomes from flipping a coin 4 times.

   (a) List all the outcomes in Ω. How many are there?

   (b) Let A be the event that the first flip is a Heads. List all the outcomes in A. How many are there?

   (c) Let B be the event that the third flip is a Heads. List all the outcomes in B. How many are there?

   (d) Let C be the event that the first flip and the third flip are both Heads. List all the outcomes in C. How many are there?

   (e) Let D be the event that the first flip or the third flip is a Heads. List all the outcomes in D. How many are there?

   (f) Are the events A and B disjoint? Express the event C in terms of A and B. Express the event D in terms of A and B.

   (g) Suppose now the coin is flipped $n \geq 3$ times instead of 4 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

4. (12 pts.)   Probability Models
Suppose you have two coins, one is biased with a probability of p coming up Heads, and one is biased with a probability of q coming up Heads. Answer the questions below, but you don't need to provide justifications.

(a) Suppose $p = 1$ and $q = 0$.

    (i) You pick one of the two coins randomly and flip it. You repeat this process n times, each time randomly picking one of the two coins and then flipping it. Consider the sample space Ω of all possible length n sequences of Heads and Tails so generated. Give a reasonable probability assignment (i.e. assign probabilities to all the outcomes) to model the situation.

    (ii) Now you pick one of the two coins randomly, but flip the *same* coin n times. Identify the sample space for this experiment together with a reasonable probability assignment to model the situation. Is your answer the same as in the previous part?

(b) Repeat the above two questions for arbitrary values of p and q. Express your answers in terms of p and q.

5. **(12 pts.) Beat Watson**
On the first day of lecture, Amir Kamil mentioned the following puzzle:

> You've been booked to play a Jeopardy tournament where to win the tournament, you have to win two *consecutive* games of Jeopardy, out of three games. You have the choice of playing Amir, then Watson, then Amir (AWA)—or Watson, then Amir, then Watson (WAW). Amir is a lousy Jeopardy player; Watson is very difficult to beat. Which schedule should you choose, to maximize your chances of winning the tournament?

(If you win the first and third game of Jeopardy but lose the second, you *lose* the tournament!) Let's analyze this puzzle. Suppose you have probability p of beating Watson in any given game, and probability q of beating Amir in any single game. Assume all three games are independent, draws never happen, and $0 < p < q < 1$.

(a) What is the probability that you win the tournament, if you choose the AWA schedule?

(b) What is the probability that you win the tournament, if you choose the WAW schedule?

(c) Which schedule offers you the better chances of winning the tournament? Does your answer depend upon the specific values of p, q? Prove your answer.

6. **(10 pts.) A fun game**
Consider a game in which you have two quarters and a table with a row of squares marked like this:
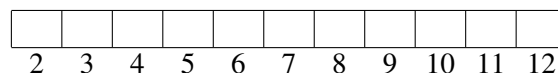
Before the game begins, you get to place each quarter on one square. You can put either both quarters on the same square, or you can put them on two different squares: your choice.

Then, you roll two fair dice, sum up the numbers showing on the dice to get a number from 2–12, and if there's a quarter on the square labelled with that number, remove it from the table. (If there are two quarters on that square, remove only one of them.) Now roll the two fair dice a second time, again getting a number from 2–12, and again removing a single quarter from the square with that number, if there's a quarter there. At this point, the game is over. If you removed both quarters, you win; if any quarter remains on the table, you lose.

(a) What's the probability of winning, if you put two quarters on the square labelled 5?

(b) What's your best strategy? In other words, what's the best place to put your two quarters, if you want to maximize the probability of winning? State where you should put your two quarters. Then, calculate the probability that you win, if you put your two quarters there.

Be careful! This one is a little tricky. You don't have to prove your answer correct on your homework solution, but you might want to do it on scratch paper for your own sake anyway.

7. **(10 pts.) Conditional probability**

(a) I have a bag containing either a \$1 or \$5 bill (with probability $1/2$ for each of these two possibilities). I then add a \$1 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag (without looking). Suppose it turns out to be a \$1 bill. If a second student draws the remaining bill from the bag, what is the chance that it too is a \$1 bill? Show your calculations.

(b) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work.

8. **(10 pts.) Poisoned pancakes**
You have been hired as an actuary by IHOP corporate headquarters, and have been handed a report from Corporate Intelligence that indicates that a covert team of ninjas hired by Denny's will sneak into some IHOP, and will have time to poison five of the pancakes being prepared (they can't stay any longer to avoid being discovered by Pancake Security). Given that an IHOP kitchen has 50 pancakes being prepared, and there are ten patrons, each ordering five pancakes (which are chosen uniformly at random from the pancakes in the kitchen), calculate the probabilities that a the first patron:

(a) will not receive any poisoned pancakes;

(b) will receive exactly one poisoned pancake;

(c) will receive at least one poisoned pancake;

(d) will receive at least one poisoned pancake given that the second patron received at least one poisoned pancake;

(e) that any of the first three receive at least one poisoned pancake.

9. **(15 pts.) Colorful coins**
We are given three coins. The first coin is a fair coin painted blue on the heads side and white on the tails side. The other two coins are biased so that the probability of heads is p. They are painted blue on the tails side and red on the heads side. One coin is randomly chosen and flipped twice.

(a) Describe the outcomes in the sample space, and give their probabilities. [NOTE: You may want to draw a tree to illustrate the sample space.]

(b) Now suppose two coins are chosen randomly *with replacement* and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in part (a)? [NOTE: You may want to draw a tree to illustrate the sample space.]

(c) Now suppose two coins are chosen randomly *without replacement* and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in parts (a) or (b)? [NOTE: You may want to draw a tree to illustrate the sample space.]

(d) Suppose the probability that the two sides that land face up are the same color is $\frac{29}{96}$ in the experiment in part (c). What does this tell you about the possible values of p?

(e) Let A be the event that you get a head on the first flip and B is the event that you get a head on the second flip. In each of the experiments in (a), (b) and (c), verify if A and B are independent events.