CS 70 Discrete Mathematics and Probability Theory
Summer 2011 Kamil HW 2

Due Tuesday, June 28, 4:59pm

Recall that unless otherwise stated, you must show all your work. Also keep in mind that you will be graded on both correctness and format, so make sure to follow the proper proof formats as demonstrated in the course reader.

1. (8 pts.) Grade these answers
You be the grader. Students have submitted the following answers to several exam questions. Assign each student answer either an A (correct yes/no answer, valid justification), a D (correct yes/no answer, invalid justification), or an F (incorrect answer). As always, $\pi = 3.14159\ldots$

(a) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
Student answer: Yes. $2\pi = 6.283\ldots$, which is less than 100. Also $\pi = 3.14159\ldots$ is less than 50. Therefore the proposition is of the form True $\implies$ True, which is true.

(b) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
Student answer: Yes. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$.

(c) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 49$. Explain your answer.
Student answer: No. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$, which does not imply $\pi < 49$.

(d) **Exam question:** Is the following proposition true? $\pi^2 < 5 \implies \pi < 5$. Explain your answer.
Student answer: No, it is false. $\pi^2 = 9.87\ldots$, which is not less than 5, so the premise is false. You can't start from a faulty premise.

2. (15 pts.) Practice with proofs
Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

(a) For all natural numbers n, if n is odd then $n^2 + 2010$ is odd.
(b) For all natural numbers n, $n^2 + 5n + 1$ is odd.
(c) For all real numbers a, b, if $a + b \geq 2010$ then $a \geq 1005$ or $b \geq 1005$.
(d) For all real numbers r, if r is irrational then $3r$ is irrational.
(e) For all natural numbers n, $10n^2 > n!$.

3. (12 pts.) Check for truth value
Let $P(n)$ denote the claim that $1 + 2 + \cdots + n \leq 2010\, n$, $Q(n)$ denote the claim that $1 + 2 + \cdots + n \leq (n^2 - 2)/2$, and $R(n)$ denote the claim that $1 + 2 + \cdots + n \leq (n+1)^2/2$. For each proposition in parts 1–6 below, say whether the proposition is true or false. In parts 4–6, prove your answer (you do not need to prove your answer to parts 1–3).

(a) $(\forall n \in \mathbb{N})(P(n))$.

(b) $(\forall n \in \mathbb{N})(Q(n))$.

(c) $(\forall n \in \mathbb{N})(R(n))$.

(d) $(\forall n \in \mathbb{N})(P(n) \implies P(n+1))$.

(e) $(\forall n \in \mathbb{N})(Q(n) \implies Q(n+1))$.

(f) $(\forall n \in \mathbb{N})(R(n) \implies R(n+1))$.

4. (14 pts.) Practice with proof by induction

Prove each of the following statements using induction on n.

(a) For $n \in \mathbb{N}$ with $n \geq 2$, define s_n by

$$s_n = (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{n}).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

(b) For all natural numbers $n \geq 1$, prove that $1 + 2 + \cdots + n \leq \frac{1}{2}n^2 + n$.

5. (10 pts.) Grade these answers

You be the grader. Students have submitted the following proofs. Decide whether you think the proof is valid or not, and assign each student answer either an A (valid proof) or an F (invalid proof). If the proof is invalid, explain *clearly and concisely* where the logical error in the proof is, including exactly which step of the reasoning is erroneous. (If you think the proof is correct, you do not need to give any explanation.) Simply saying that the claim (or the induction hypothesis) is false is *not* an acceptable explanation.

(a) **Claim**: For every $n \in \mathbb{N}$ with $n \geq 1$, $n^2 + n$ is odd.

Proof: The proof will be by induction.

Base case: For the natural number $n = 1$, $1^2 + 1$ is odd, so the statement is true for $n = 1$.

Induction hypothesis: Assume $k^2 + k$ is odd, for some $k \in \mathbb{N}$.

Inductive step: We want to prove that $(k+1)^2 + (k+1)$ is odd. Calculate:

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + (2k + 2),$$

which is the sum of an odd and an even integer. Therefore, $(k+1)^2 + (k+1)$ is odd. By the principle of mathematical induction, the property that $n^2 + n$ is odd is true for all natural numbers n. □

(b) **Claim:** For all natural numbers $n \geq 4$, $2^n < n!$.

Proof: Base case: $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$.

Induction hypothesis: Assume that $2^k < k!$, for some $k \in \mathbb{N}$.

Inductive step: We will show that $2^{k+1} < (k+1)!$. Applying the induction hypothesis, we see that

$$2^{k+1} = 2 \times 2^k < 2 \times k! \leq (k+1) \times k! = (k+1)!,$$

so $2^{k+1} < (k+1)!$.

Therefore, by the principle of mathematical induction, the statement is true for all $n \geq 4$. □

(c) **Claim:** For every $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Proof: The proof will be by induction on n.

Base case: For $n = 0$, $\sum_{i=0}^{0} 2^i = 2^0 = 1$ and $2^{0+1} - 1 = 2^1 - 1 = 1$, so the claim holds for $n = 0$.

Inductive step: By the induction hypothesis, we have

$$\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1.$$

Breaking out the last term of the sum on the left and using the fact that $2^{n+2} = 2^{n+1} + 2^{n+1}$, we get

$$2^{n+1} + \sum_{i=0}^{n} 2^i = 2^{n+1} + 2^{n+1} - 1.$$

Now subtracting 2^{n+1} from both sides yields

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

This is what we needed to prove, so the claim is true for all n. $\square$

(d) **Claim:** For every $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Proof: The proof will be by induction on n.

Base case: For $n = 0$, $\sum_{i=0}^{0} 2^i = 2^0 = 1$ and $2^{0+1} - 1 = 2^1 - 1 = 1$, so the claim holds for $n = 0$.

Induction hypothesis: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.

Inductive step: We need to prove

$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1.$$

Now we can do a little calculation, applying the induction hypothesis at just the right point:

$$\begin{aligned}\sum_{i=0}^{k+1} 2^i &= 2^{k+1} + \sum_{i=0}^{k} 2^i \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2^{k+2} - 1.\end{aligned}$$

So we have proven that the claim holds for $n = k+1$. Therefore, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$. $\square$

(e) **Claim:** For all $x, y, n \in \mathbb{N}$, if $\max(x, y) = n$, then $x = y$.

Proof: The proof of the claim will be by induction on n.

Base case: Suppose that $n = 0$. If $\max(x, y) = 0$ and $x, y \in \mathbb{N}$, then $x = 0$ and $y = 0$, hence $x = y$.

Induction hypothesis: Assume that, whenever we have $\max(x, y) = k$, then $x = y$ must follow.

Inductive step: We must prove that if $\max(x, y) = k + 1$, then $x = y$. Suppose x, y are such that $\max(x, y) = k + 1$. Then it follows that $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. In this case, adding 1 to both sides, we have $x = y$, completing the induction step.

Therefore, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$. $\square$

6. (12 pts.) Principle of induction

Let $P(k)$ be a proposition involving a natural number k. Suppose you know only that $(\forall k \in \mathbb{N})(P(k) \implies P(k+2))$ is true. For each of the following propositions, say whether the proposition is (i) definitely true, (ii) definitely false, or (iii) possibly (but not necessarily) true. Give a *brief* (one or two sentences) explanation for each of your answers.

(a) $(\forall n \in \mathbb{N})(P(n))$.
(b) $(\forall n \in \mathbb{N})(\neg P(n))$.
(c) $P(0) \implies (\forall n \in \mathbb{N})(P(n+2))$.
(d) $(P(0) \wedge P(1)) \implies (\forall n \in \mathbb{N})(P(n))$.
(e) $(\forall n \in \mathbb{N})(P(n) \implies ((\exists m \in \mathbb{N})(m > n + 2010 \wedge P(m))))$.
(f) $(\forall n \in \mathbb{N})(n < 2010 \implies P(n)) \wedge (\forall n \in \mathbb{N})(n \geq 2010 \implies \neg P(n))$.

7. (9 pts.) Recurrence relations

Let $f(n)$ be defined by the recurrence relation $f(n) = 7f(n-1) - 10f(n-2)$ (for all $n \geq 2$) and $f(0) = 1$, $f(1) = 2$. Prove that $f(n) = 2^n$ for every $n \in \mathbb{N}$.

8. (10 pts.) Let's be social

n people go to a bar. Initially, each person sits at their own table. After a little while, the bartender picks a table, taps the person at that table on the shoulder, and asks him to move to a second table. The person who just moved introduces himself and shakes hands with the person who was already sitting at the second table.

In general, the bartender keeps repeating the following operation: the bartender chooses two tables; the bartender asks everyone sitting at the first table to move over to the second table; and each of the folks who just moved from the first table shake hands with everyone who was already sitting at the second table. Suppose that there were k people sitting at the first table and ℓ people sitting at the second table before this operation. After this operation, there are 0 people at the first table and $k+\ell$ people at the new table. Also, each of the k newcomers shakes hands with each of the ℓ folks already at the second table, so $k\ell$ handshakes occur during this operation. The bartender repeats this kind of operation until all n people are sitting at the same table.

Let $H(n)$ denote the the total number of handshakes that have occurred among the n people by the time this process is finished and everyone is seated at the same table. Prove that it doesn't matter what order the bartender decides to choose tables; we always have $H(n) = n(n-1)/2$.

Hint: Use strong induction.

9. (10 pts.) Tower of Brahma

This puzzle was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

Guess and prove what is the minimum number of moves required to carry out this task in general, if there are n disks on the original needle. Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy predict before the destruction of the World?

Hint: Whatever strategy you decide to use, what configuration must the disks be in whenever the largest disk is moved?