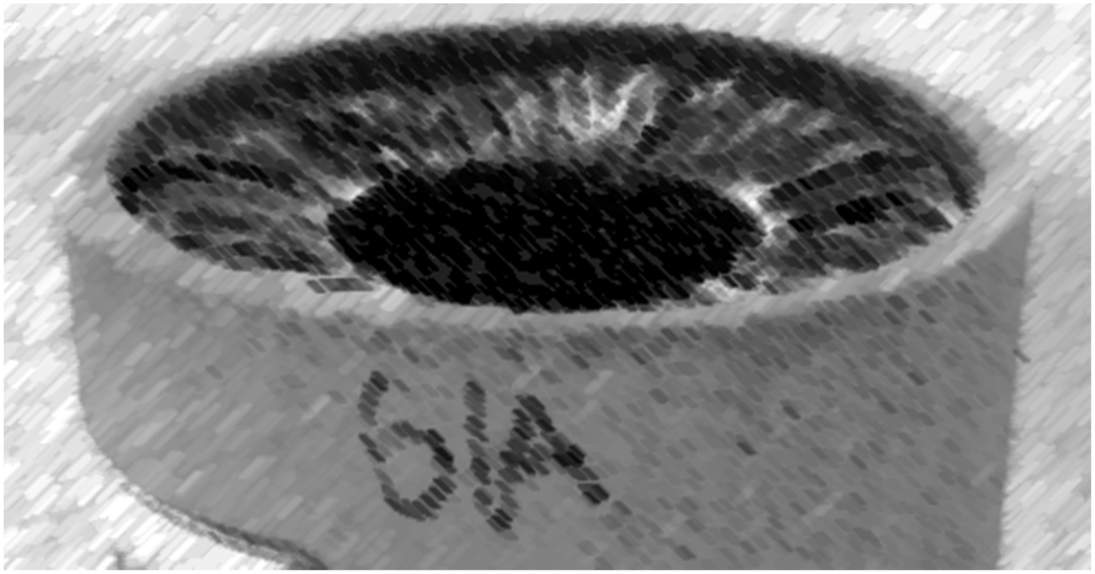

CS61A Lecture 38

Robert Huang

UC Berkeley

April 17, 2013

Announcements

- HW12 due Wednesday
- Scheme project, contest out

Review: Program Generator

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A computer program is just a sequence of bits

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from itertools import product

def bitstrings():
    size = 0
    while True:
        tuples = product(('0', '1'), repeat=size)
        for elem in tuples:
            yield ''.join(elem)
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>>> [next(bs) for _ in range(0, 10)]
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['', '0', '1', '00', '01', '10', '11', '000', '001', '010']
```

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```
[F] T   T   T   T   F   T   F   T   F   . . .
 T [T]  F   F   F   F   F   T   F   T   . . .
 T  F  [T]  F   T   F   T   F   T   T   . . .
 T  F   F  [T]  T   F   F   T   F   T   . . .
 T  F   T   T  [F]  T   F   T   F   T   . . .
 F  F   F   F   T  [F]  F   F   T   T   . . .
 T  F   T   F   F   F  [F]  T   T   T   . . .
 F  T   F   T   T   F   T  [F]  F   T   . . .
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T	[T]	F	F	F	F	F	T	F	T	. . .
T	F	[T]	F	T	F	T	F	T	T	. . .
T	F	F	[T]	T	F	F	T	F	T	. . .
T	F	T	T	[F]	T	F	T	F	T	. . .
F	F	F	F	T	[F]	F	F	T	T	. . .
T	F	T	F	F	F	[F]	T	T	T	. . .
F	T	F	T	T	F	T	[F]	F	T	. . .
T	F	T	F	F	T	T	F	[F]	T	. . .
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					. . .					

Functions

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T	F	T	T	[F]	T	F	T	F	T	. . .
F	F	F	F	T	[F]	F	F	T	T	. . .
T	F	T	F	F	F	[F]	T	T	T	. . .
F	T	F	T	T	F	T	[F]	F	T	. . .
T	F	T	F	F	T	T	F	[F]	T	. . .
F	T	T	T	T	T	T	T	T	[F]	. . .

. . .

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            T  F   F  [T]  T   F   F   T   F   T   . . .
            T  F   T   T  [F]  T   F   T   F   T   . . .
            F  F   F   F   T  [F]  F   F   T   T   . . .
            T  F   T   F   F   F  [F]  T   T   T   . . .
            F  T   F   T   T   F   T  [F]  F   T   . . .
            T  F   T   F   F   T   T   F  [F]  T   . . .
Functions   F  T   T   T   T   T   T   T   T  [F]  . . .
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def func(x):
    return (x * x) % 2
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A mathematical function *f* is *computable* if there exists a program (i.e. a Python function) **`func`** that computes it

Computability

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Are all functions computable?

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So if we have a program that computes the following function, we can run it on our programs to determine if they have infinite loops:

$$haltsonallinputs : Programs \to \{0, 1\},$$
$$haltsonallinputs(P) = \begin{cases} 1 & \text{if } P \text{ halts on all inputs} \\ 0 & \text{otherwise} \end{cases}$$

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Thus, we have to do something more clever, analyzing the program itself

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Remember, we can pass a function itself as its argument. Thus, we can consider **`halts(f, f)`**; in other words, does function **`f`** halt when given itself as an argument? (This is just a thought experiment.)

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`turing` will go into an infinite loop if **`f`** halts when given itself as an argument. Otherwise, **`turing`** returns **`True`**.

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We have a contradiction! Our assumption that **`halts`** exists is false.

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```
def is_valid_python_function(bitstring):
    """Determine whether or not a bitstring represents a
      syntactically valid 1-argument Python function."""
```

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def is_valid_python_function(bitstring):
    """Determine whether or not a bitstring represents a
      syntactically valid 1-argument Python function."""

def bitstring_to_python_function(bitstring):
    """Coerce a bitstring representation of a Python
      function to the function itself."""
```

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def function_stream():
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    bitstring_stream = iterator_to_stream(bitstrings())
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On HW12

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On HW12

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def function_stream():
    """Return a stream of all valid 1-argument Python
      functions."""
    bitstring_stream = iterator_to_stream(bitstrings())
    valid_stream = filter_stream(is_valid_python_function,
                                 bitstring_stream)
```

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    valid_stream = filter_stream(is_valid_python_function,
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    return map_stream(bitstring_to_python_function,
                      valid_stream)
```

On HW12

Filtering Out Non-Terminating Programs

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    def halt_checker(fn):
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def make_halt_checker():
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        nonlocal index
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This results in an infinite loop, which means **`halt_checker`** will return false on **`church`**, since it does not halt given its own index

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So we made a false assumption somewhere

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It is a fundamental limitation of all computers and programming languages

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Since we know we can't write **`halts`**, our assumption that we can write **`prints_something`** is false

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For example, perfect anti-virus software is impossible

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We can't write perfect security analyzers, optimizing compilers, etc.

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Thus, if a valid proof exists for a mathematical formula, then a computer can find it

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Thus, there must be values of *P* and *n* for which *H(P, n)* is neither provable nor disprovable, or for which an incorrect result can be proven

Thus, there are fundamental limitations not only to computation, but to mathematics itself!

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`os.system('python <file>')`: Directs the operating system to invoke a new instance of the Python interpreter.