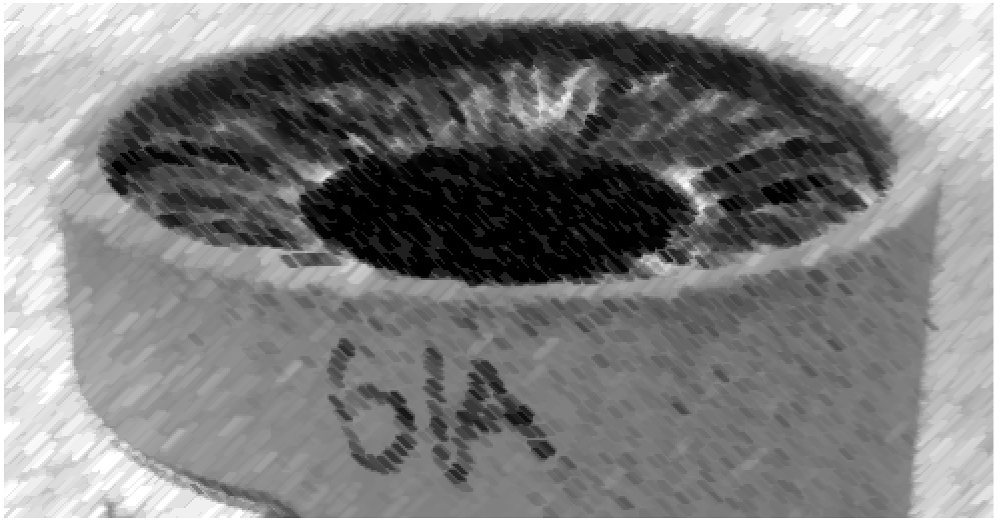

CS61A Lecture 11

Amir Kamil
UC Berkeley
February 15, 2013

Announcements

- HW4 due Wednesday at 11:59pm
- Hog contest deadline next week
  - Completely optional, opportunity for extra credit
  - See website for details

Fibonacci Sequence

The Fibonacci sequence is defined as

$$\mathrm{fib}(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ \mathrm{fib}(n-1) + \mathrm{fib}(n-2), & n > 1 \end{cases}$$

Example: http://goo.gl/DZbRG

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def fib(n):
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```

Two recursive calls!

Example: http://goo.gl/DZbRG

Tree recursion

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Executing the body of a function may entail more than one recursive call to that function

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fib(5)

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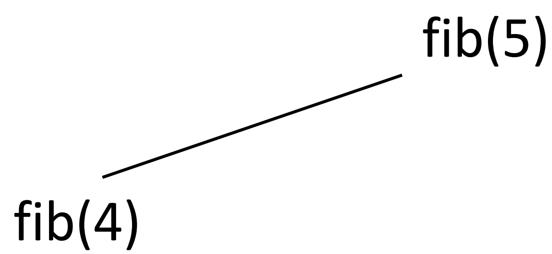

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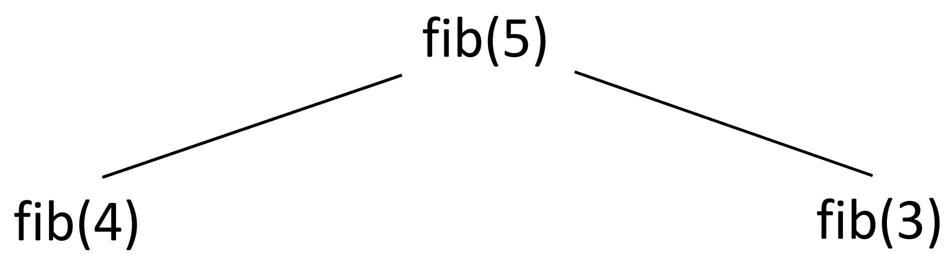

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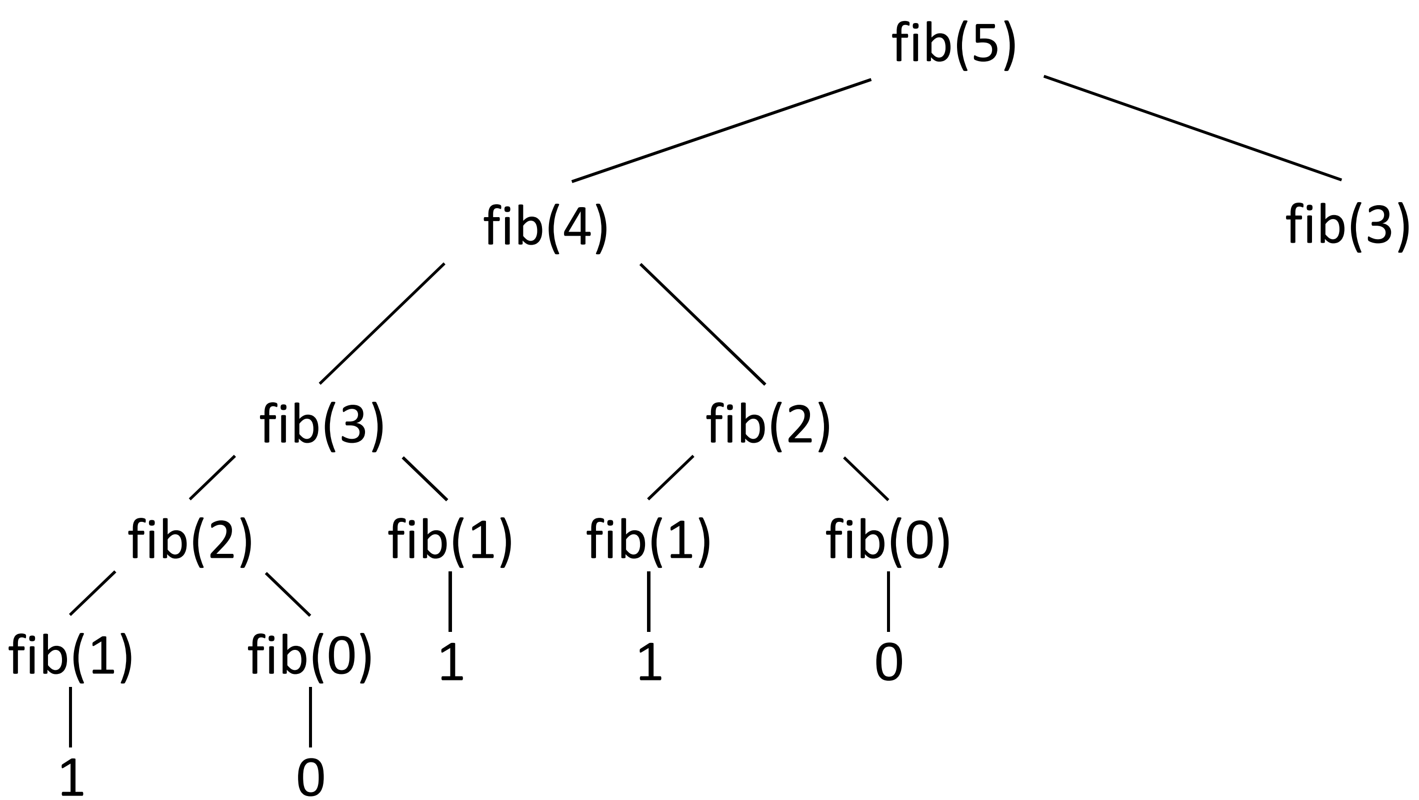

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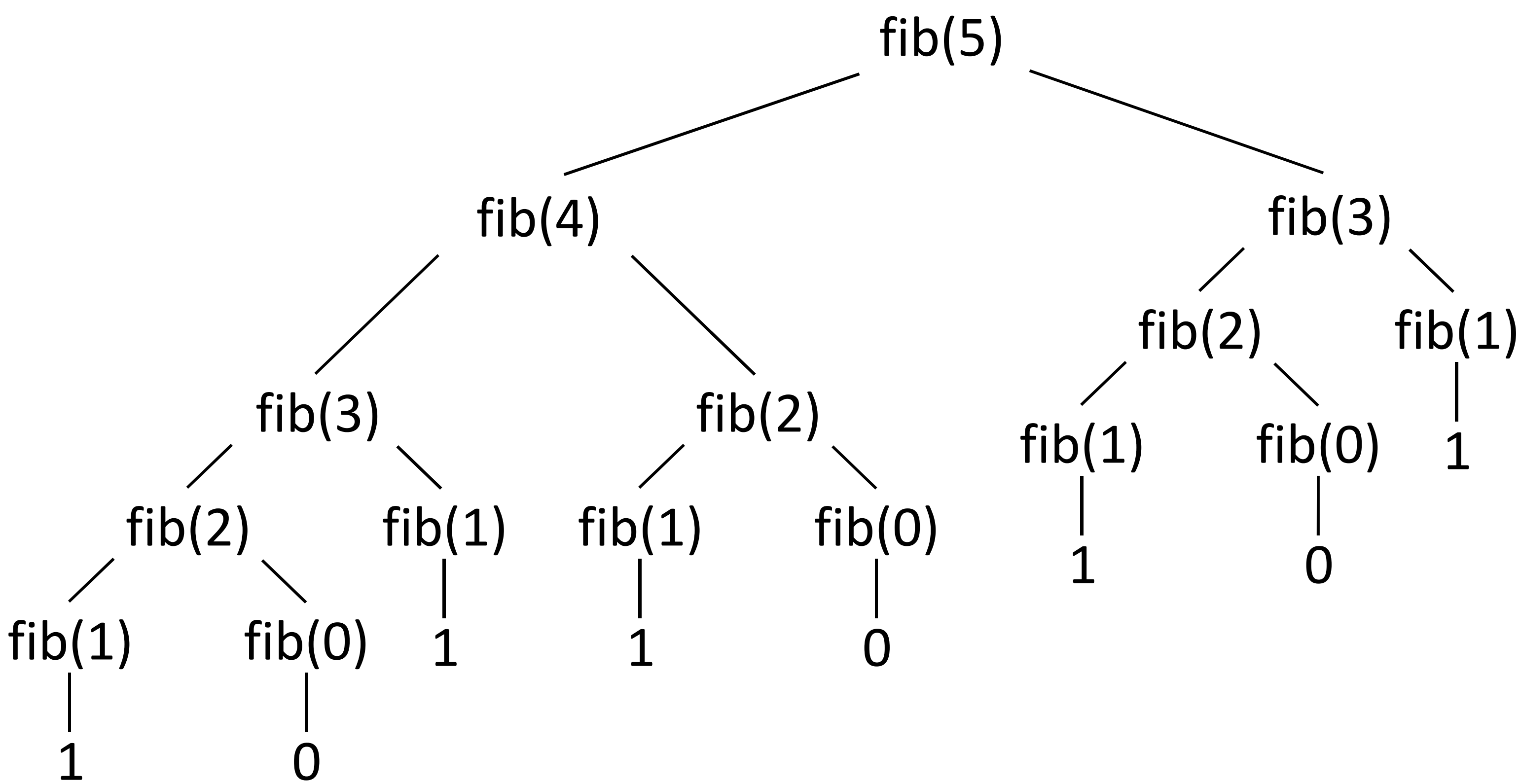

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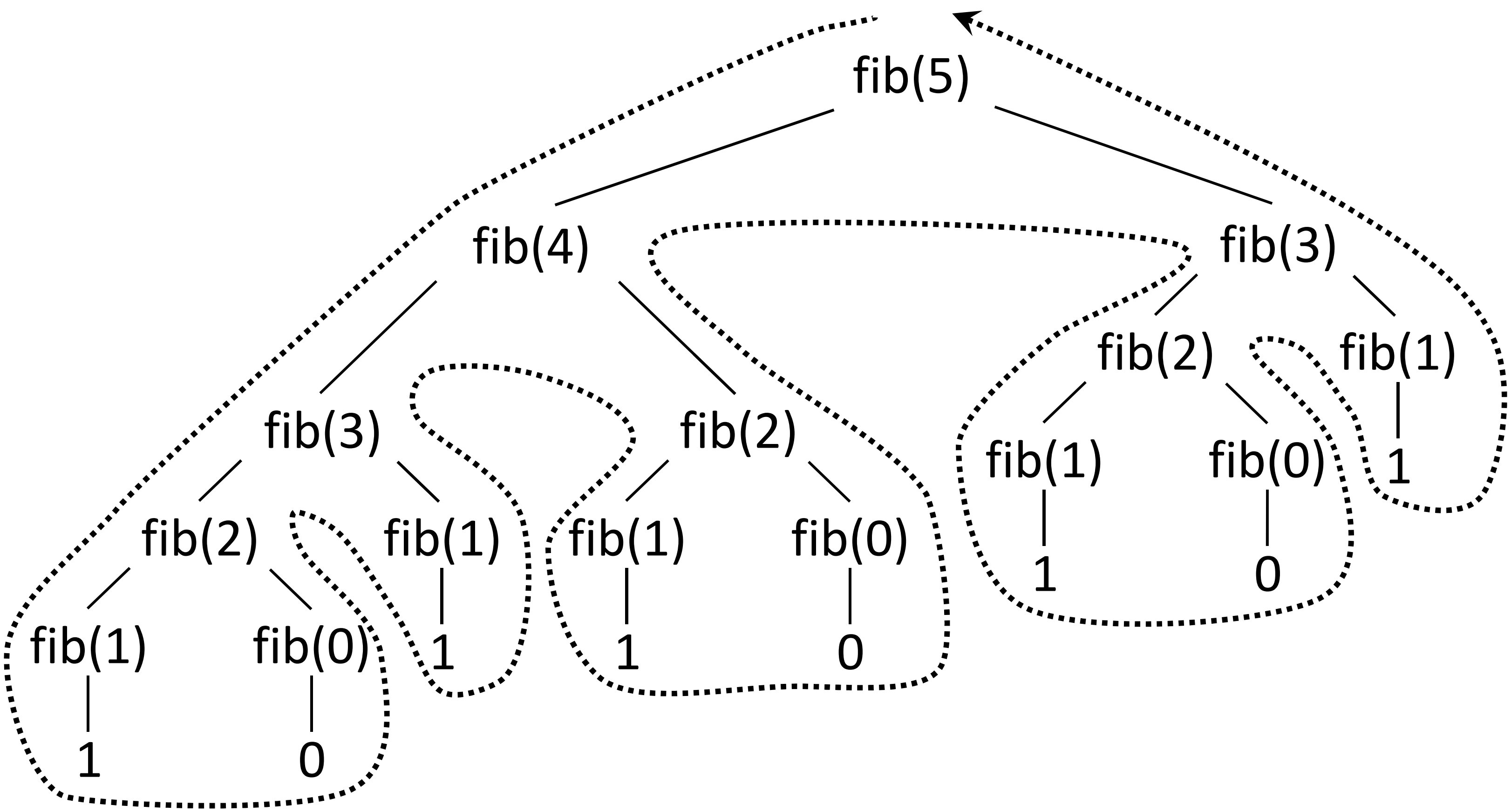

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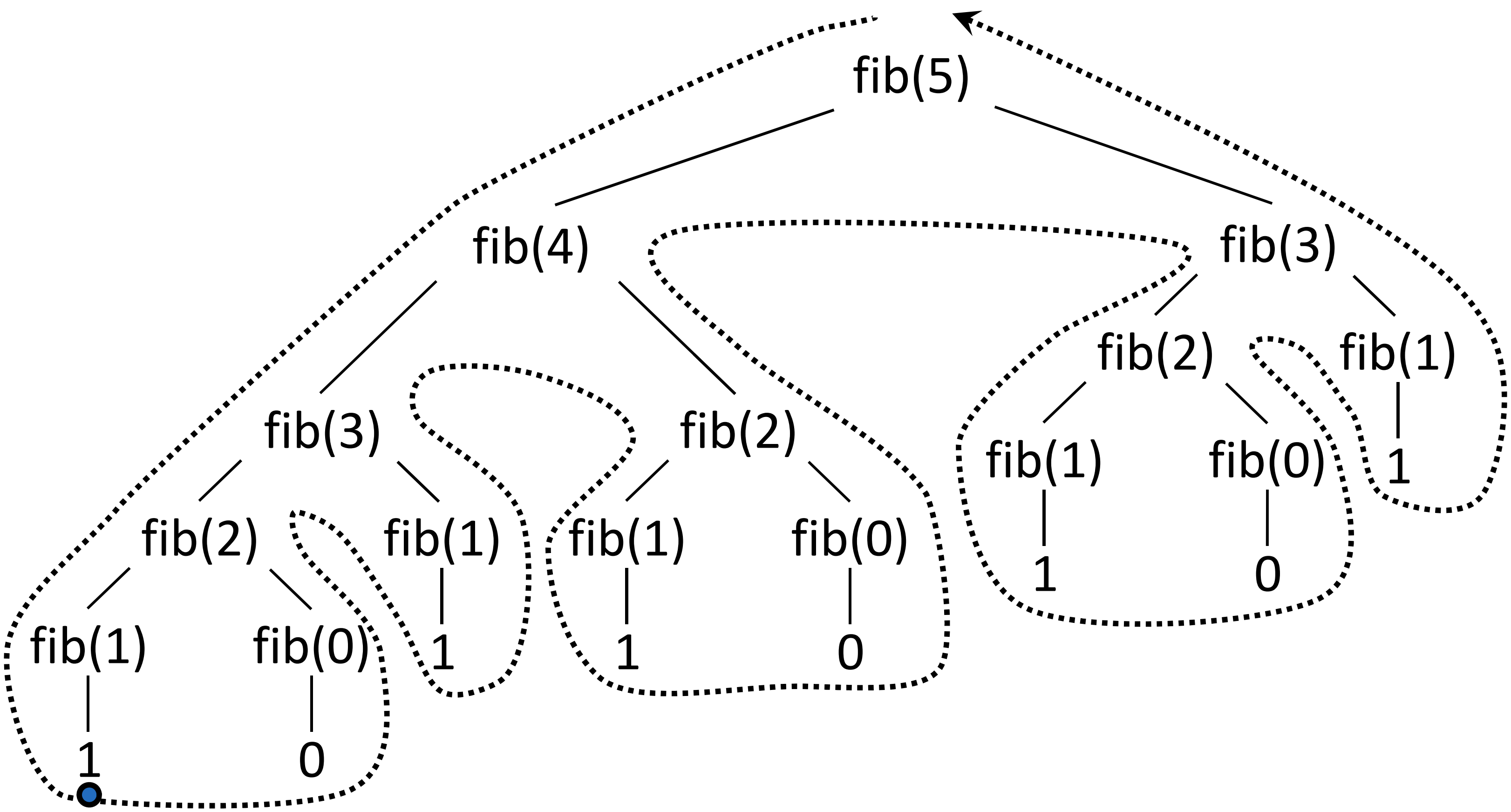

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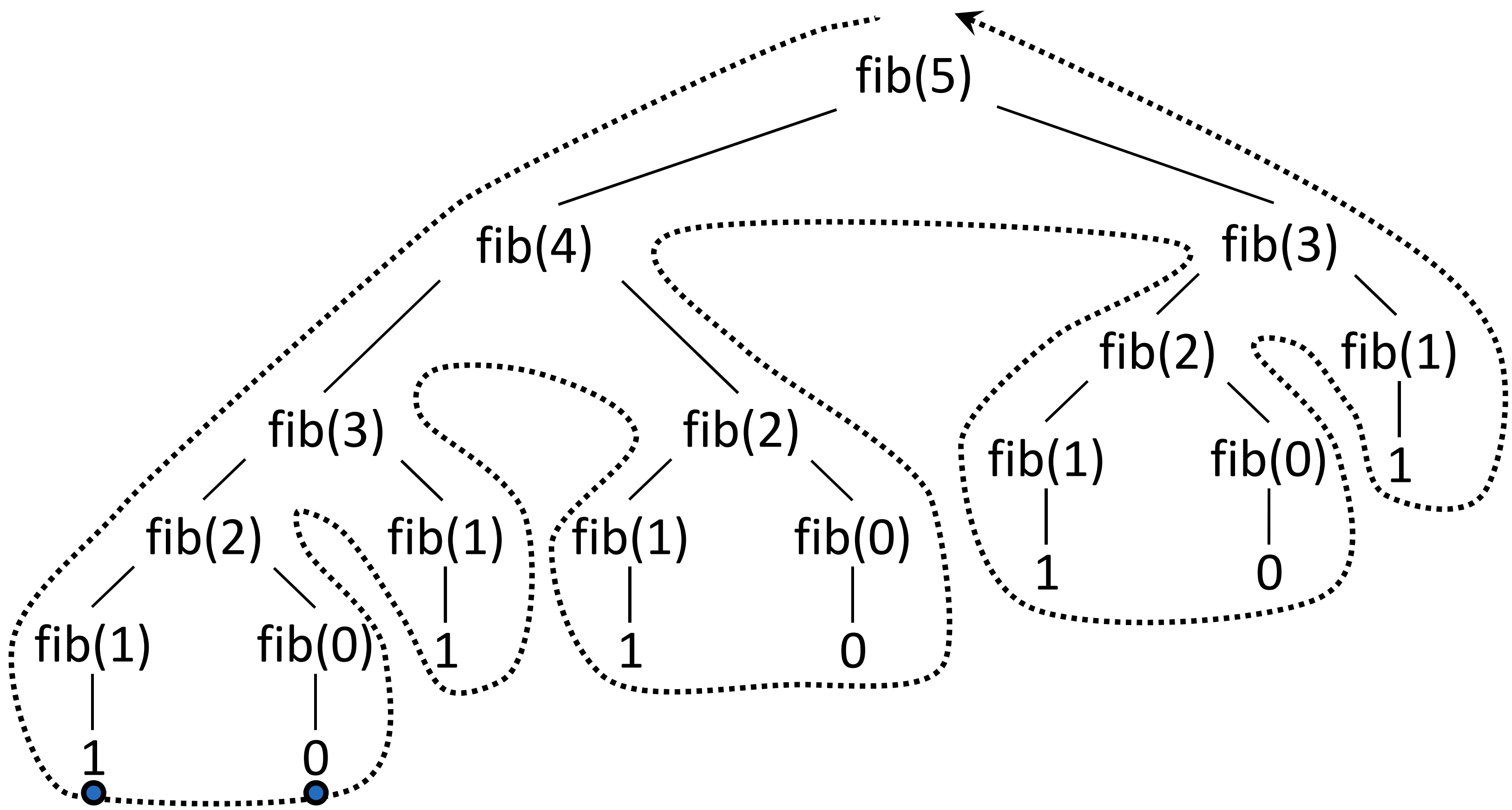

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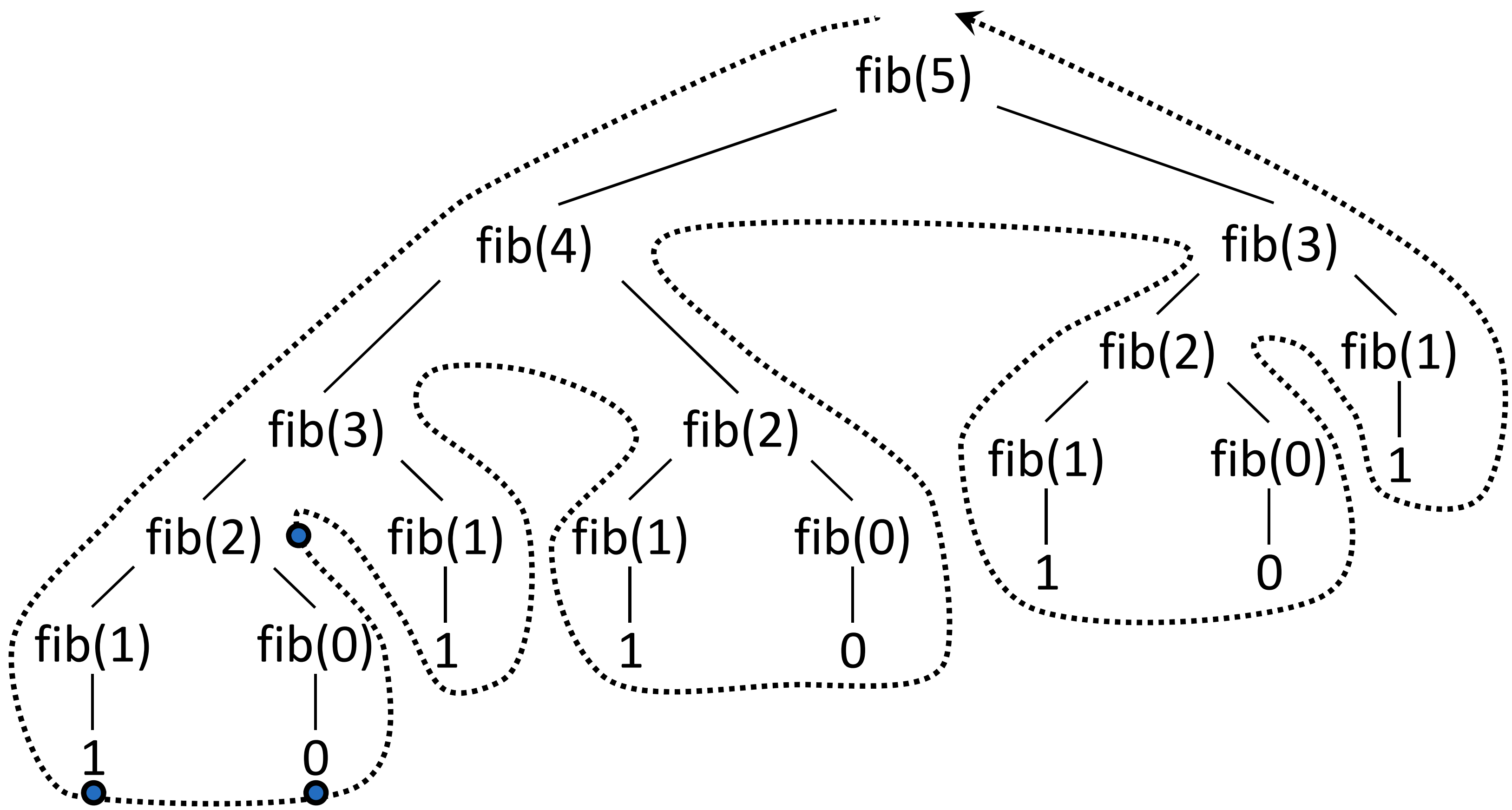

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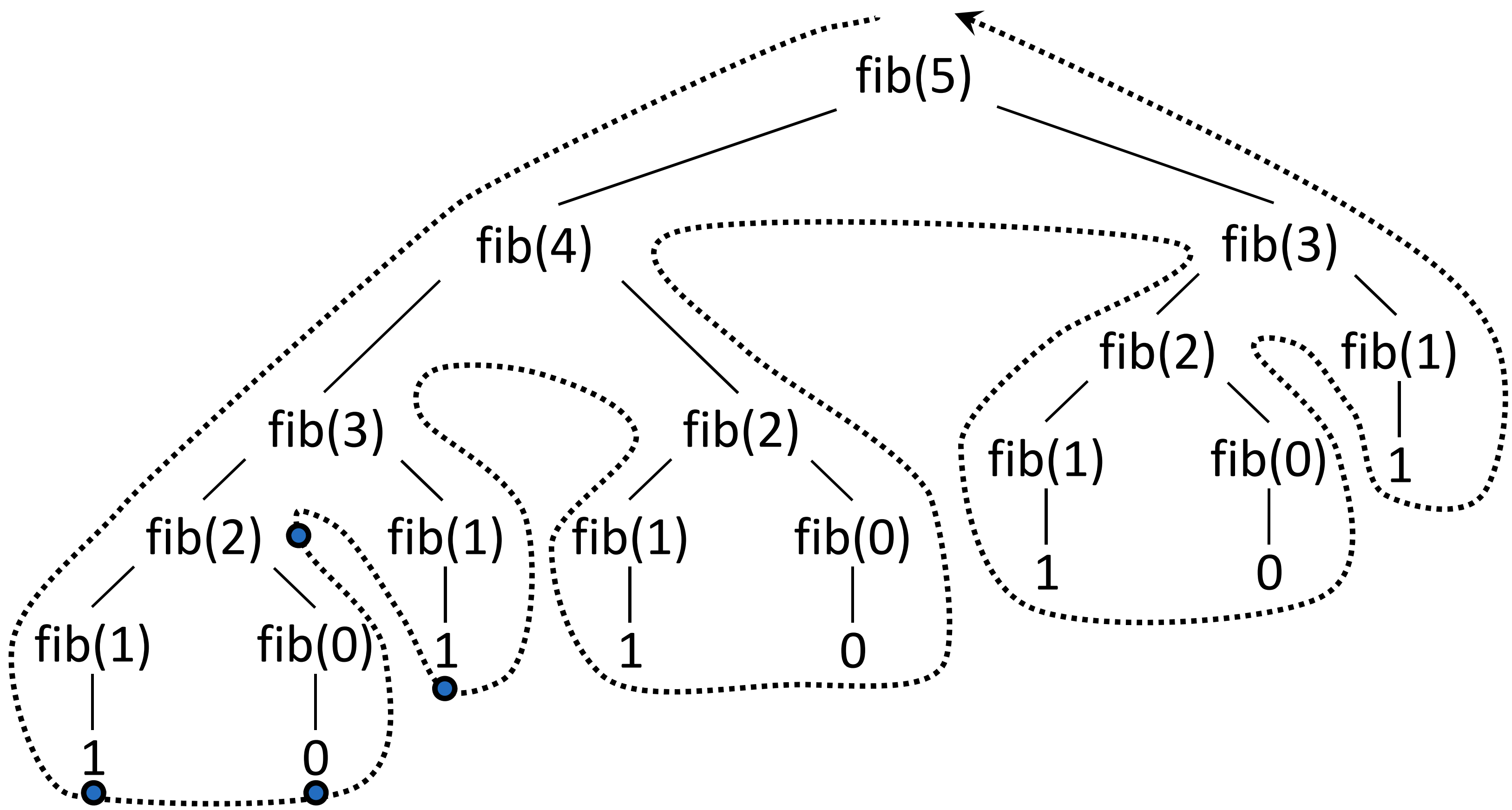

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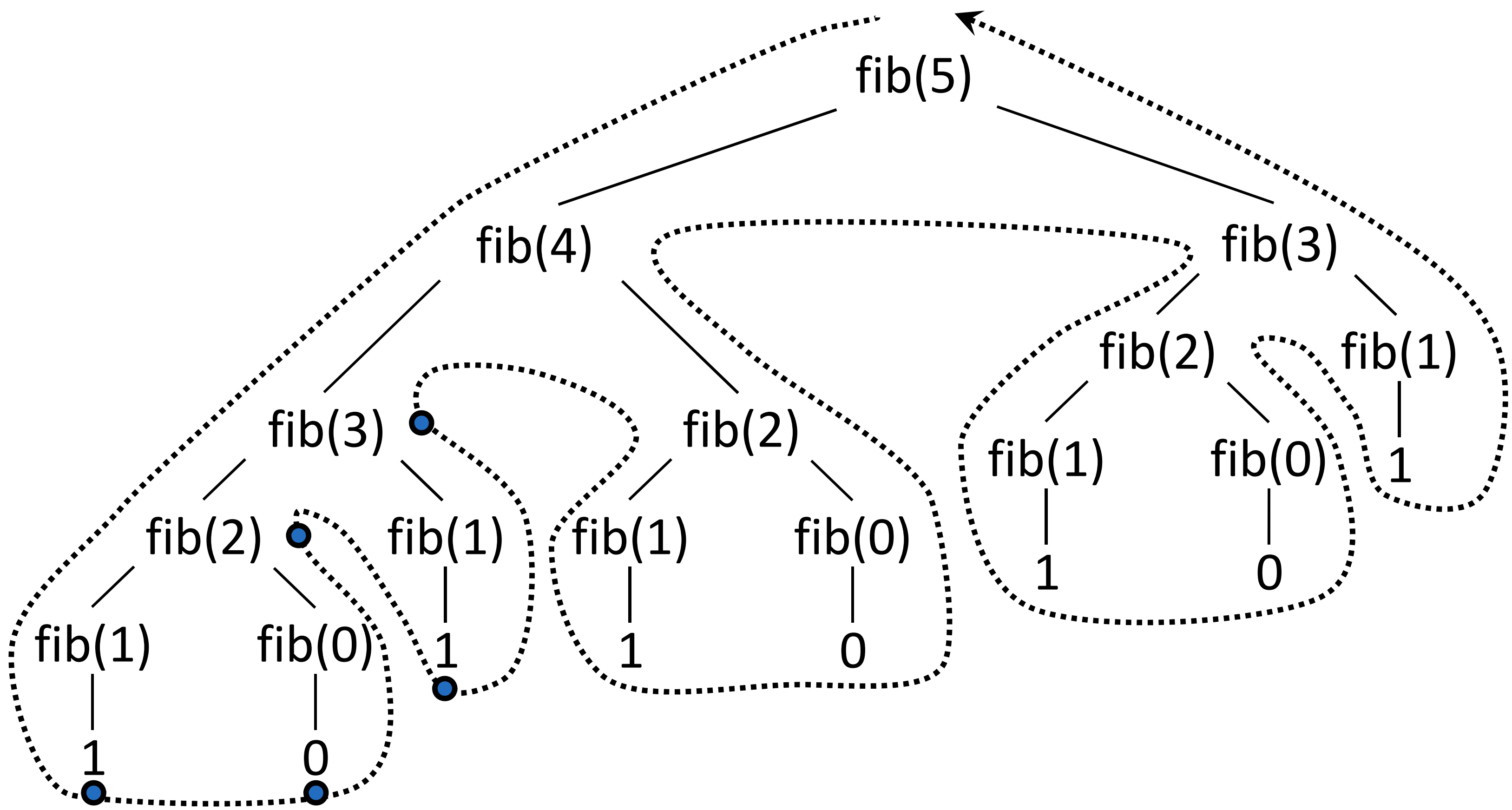

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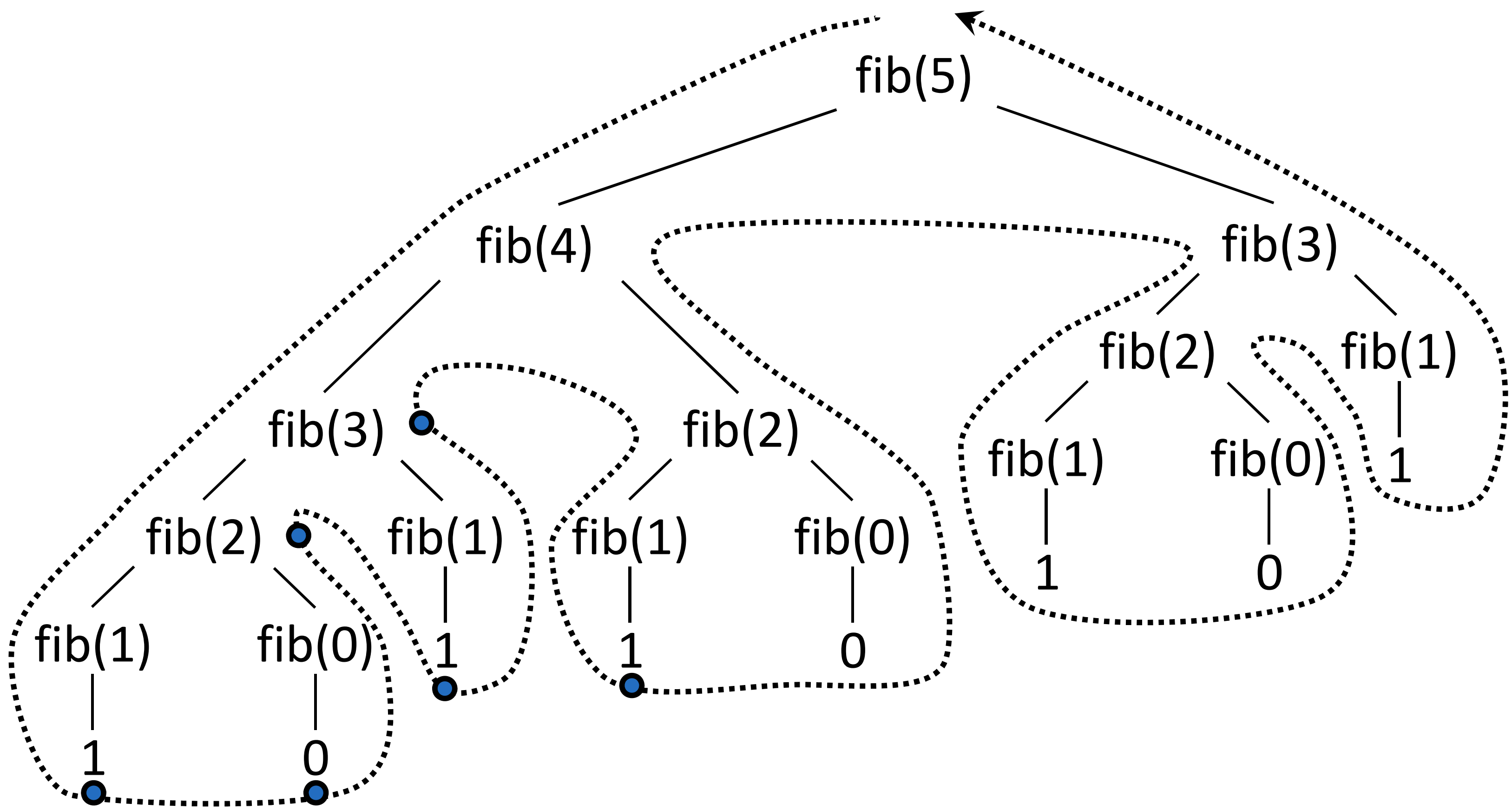

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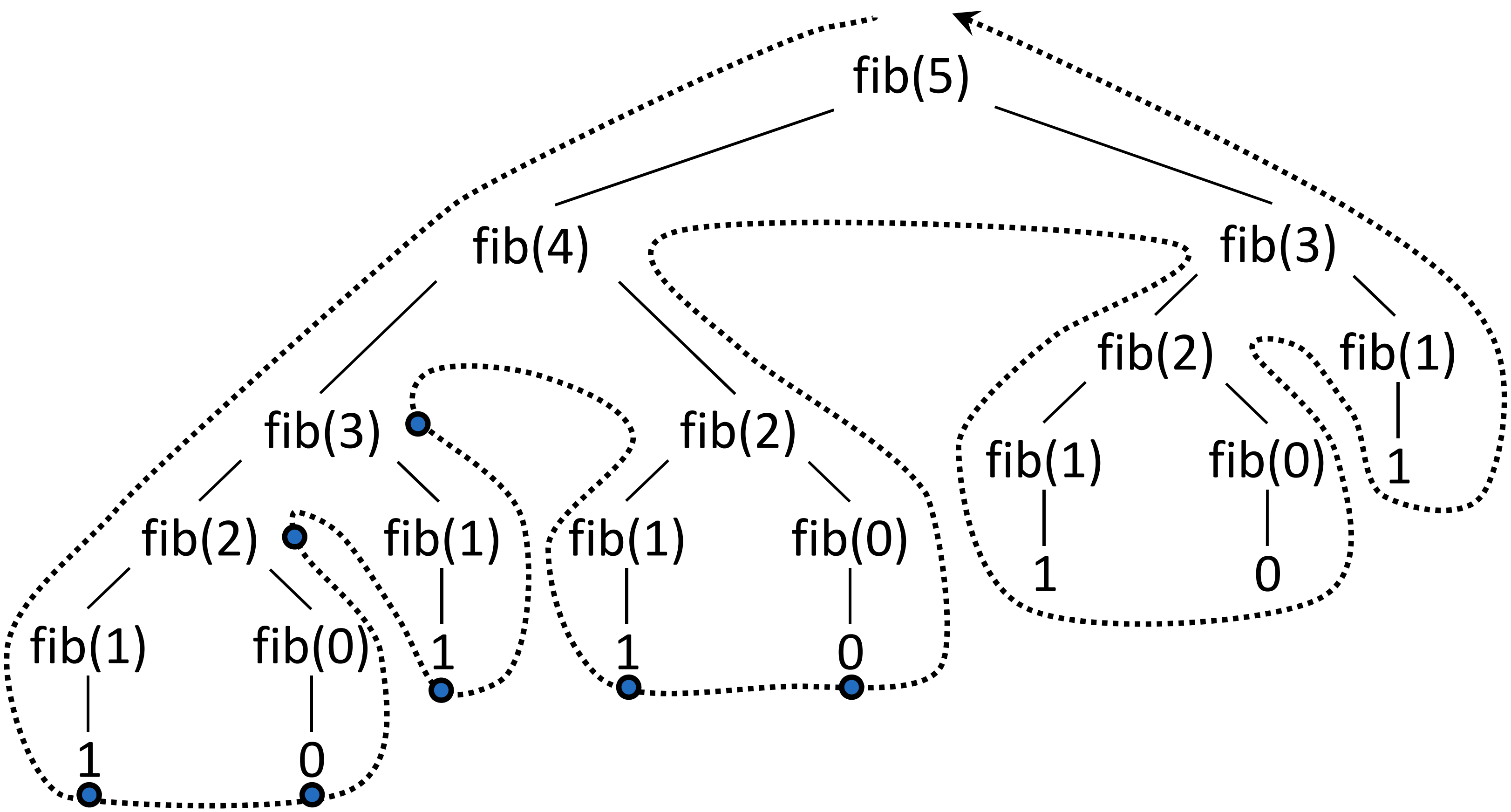

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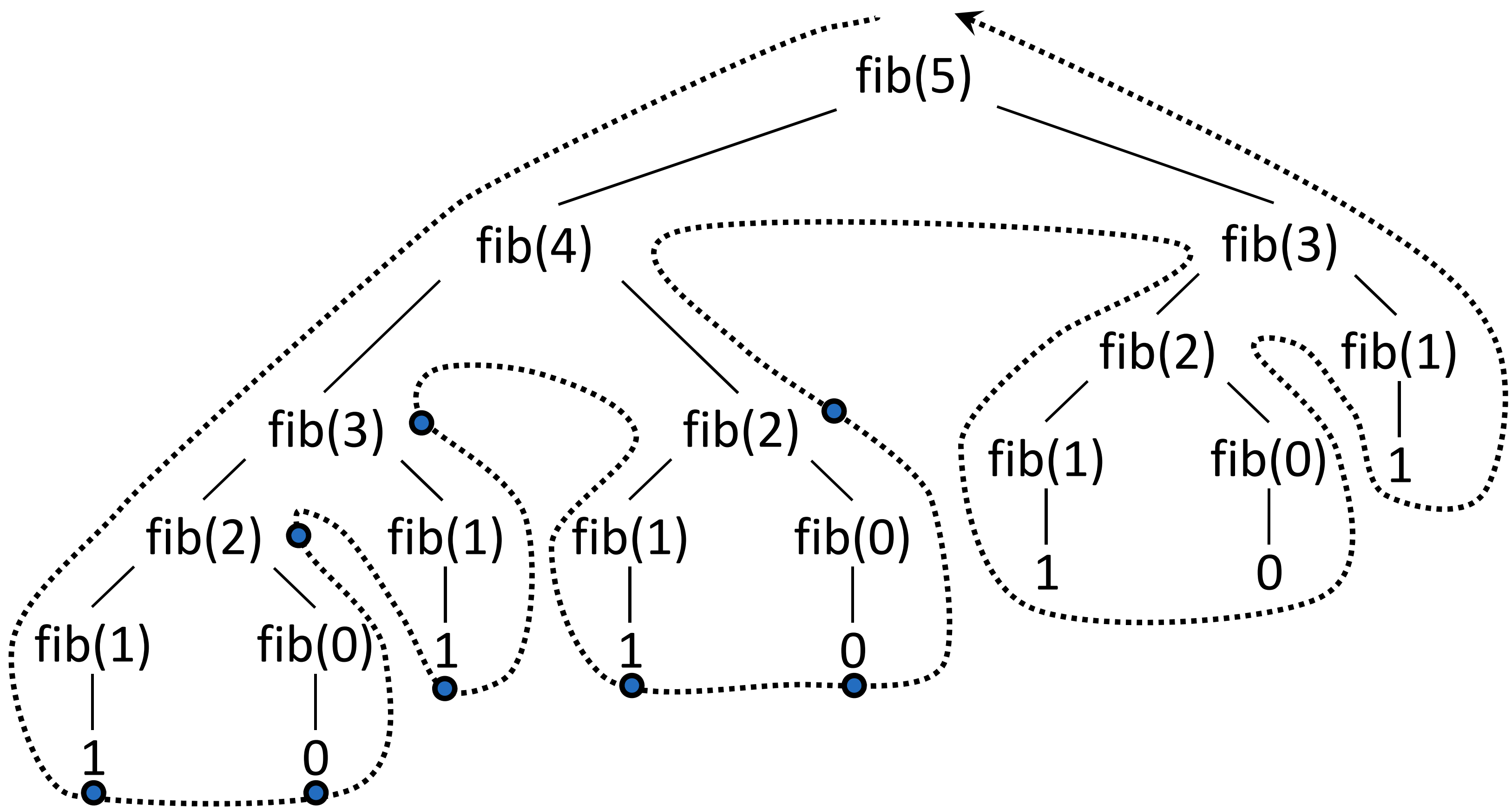

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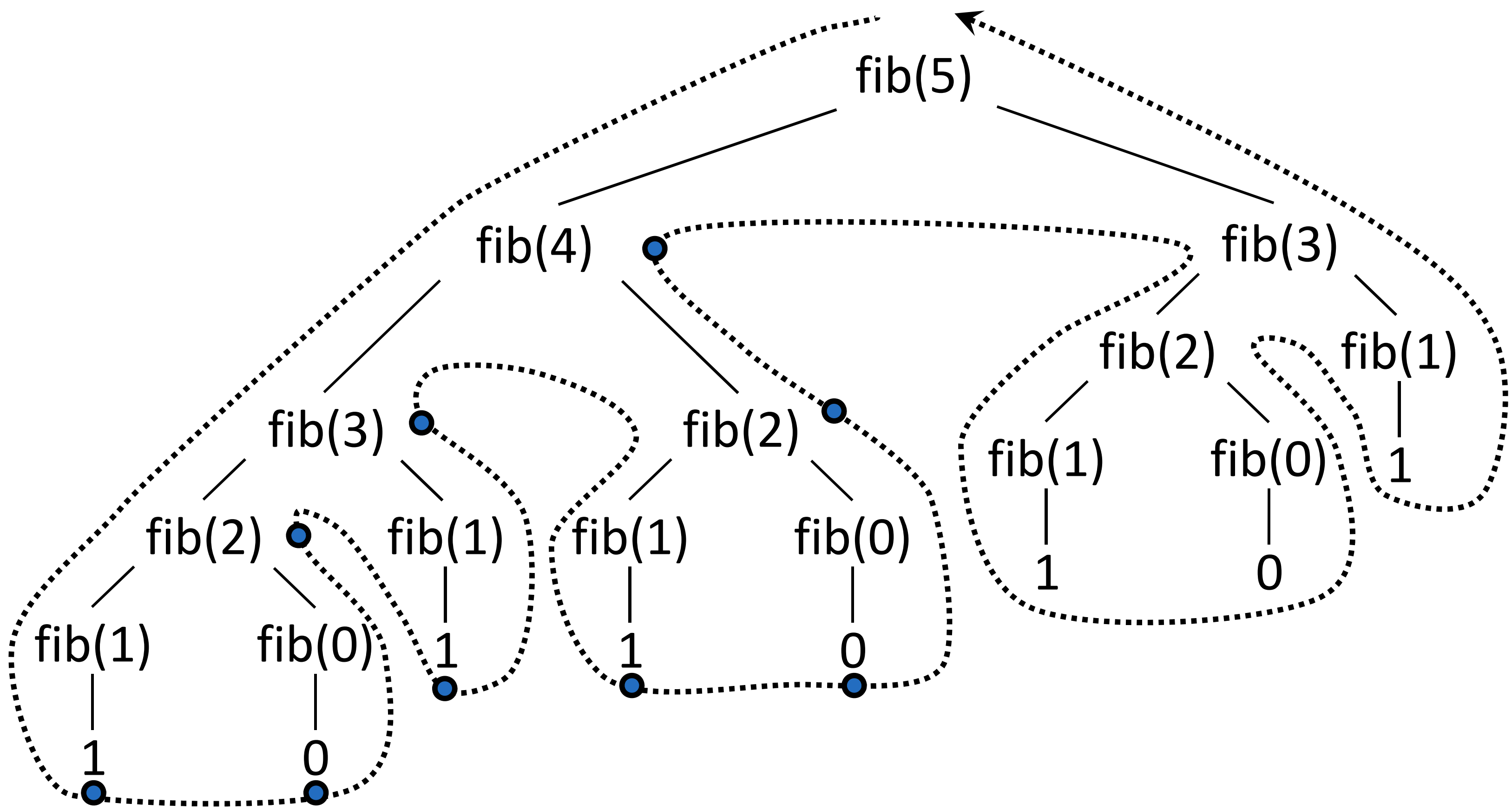

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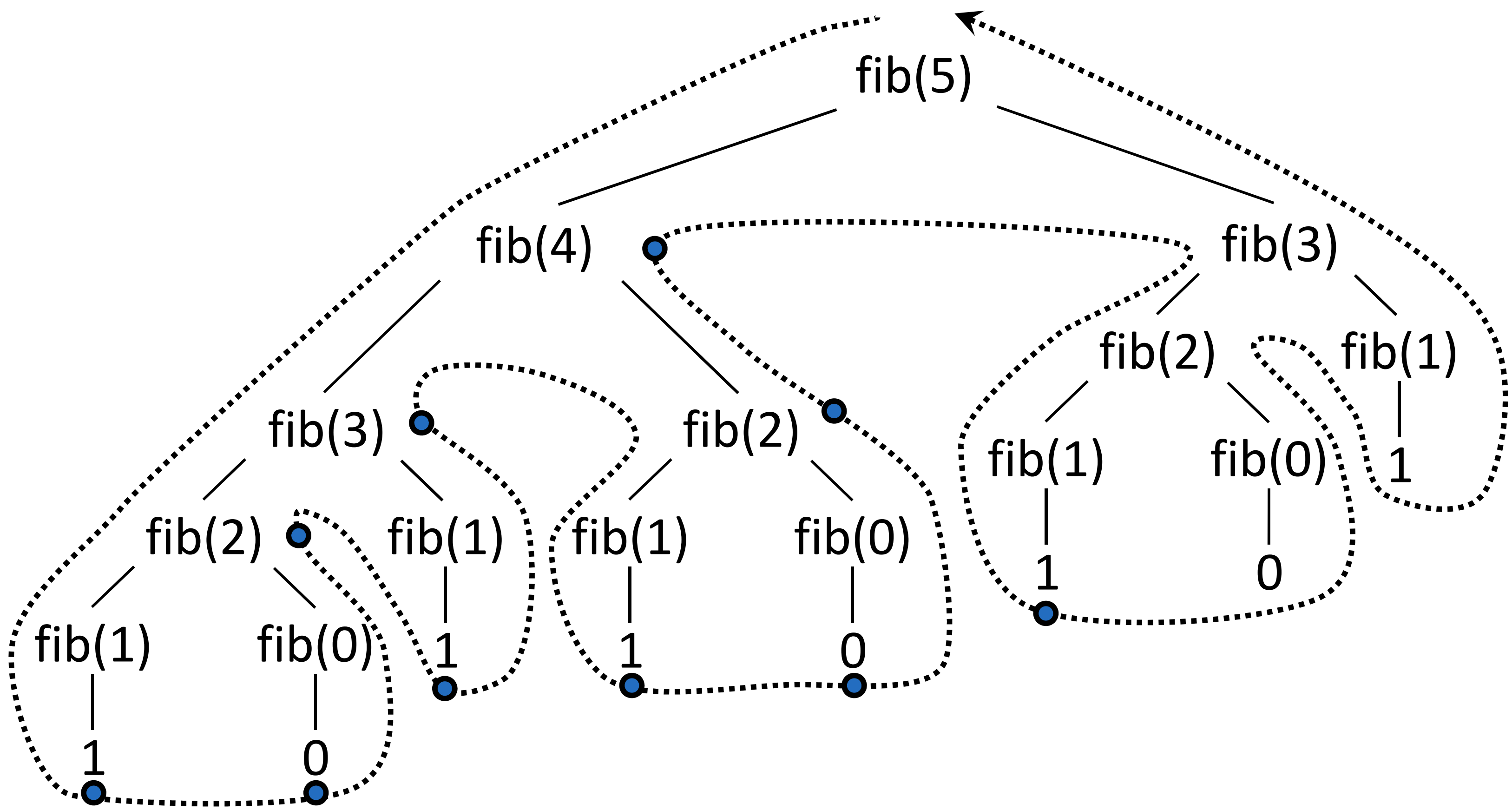

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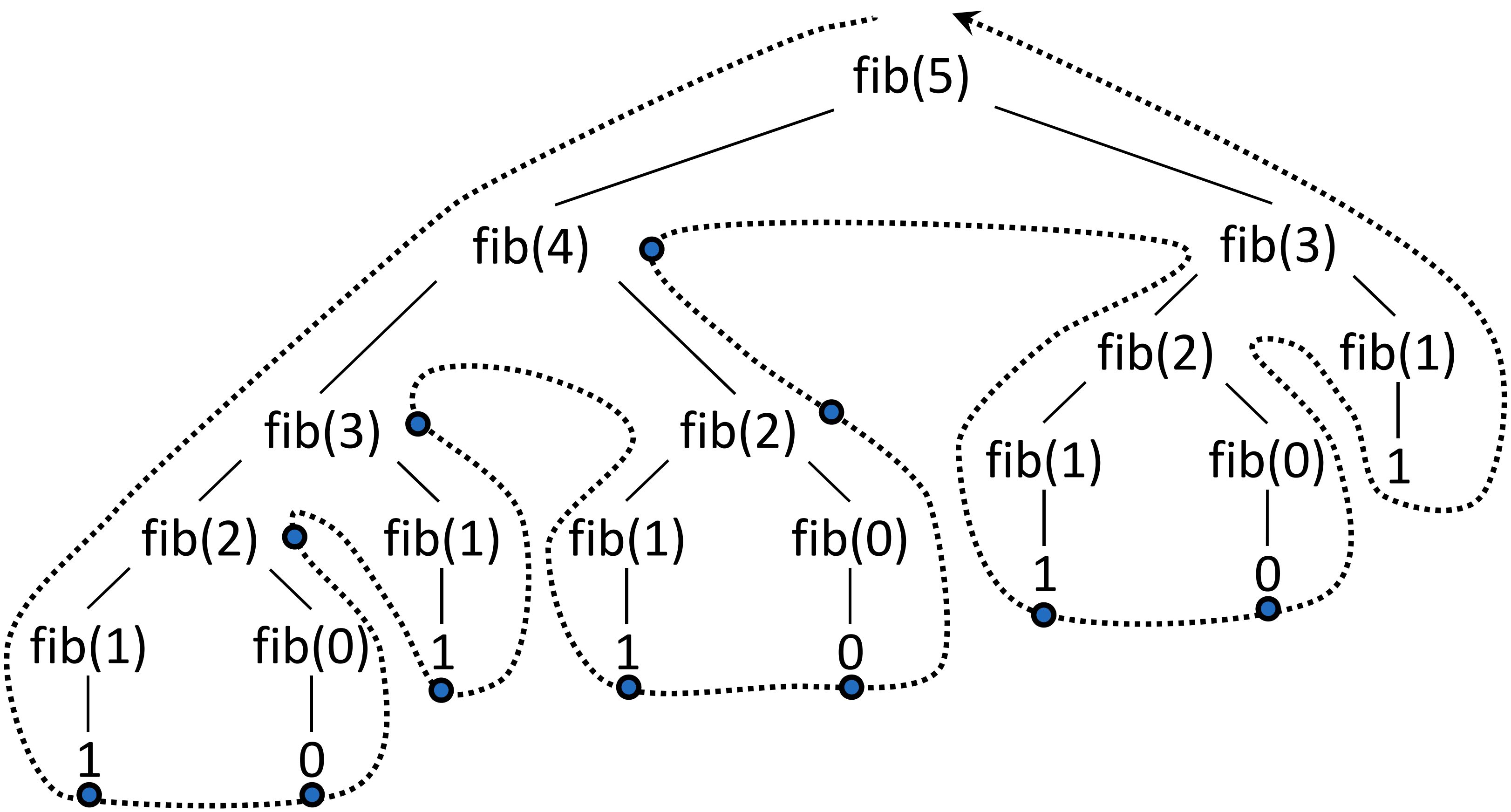

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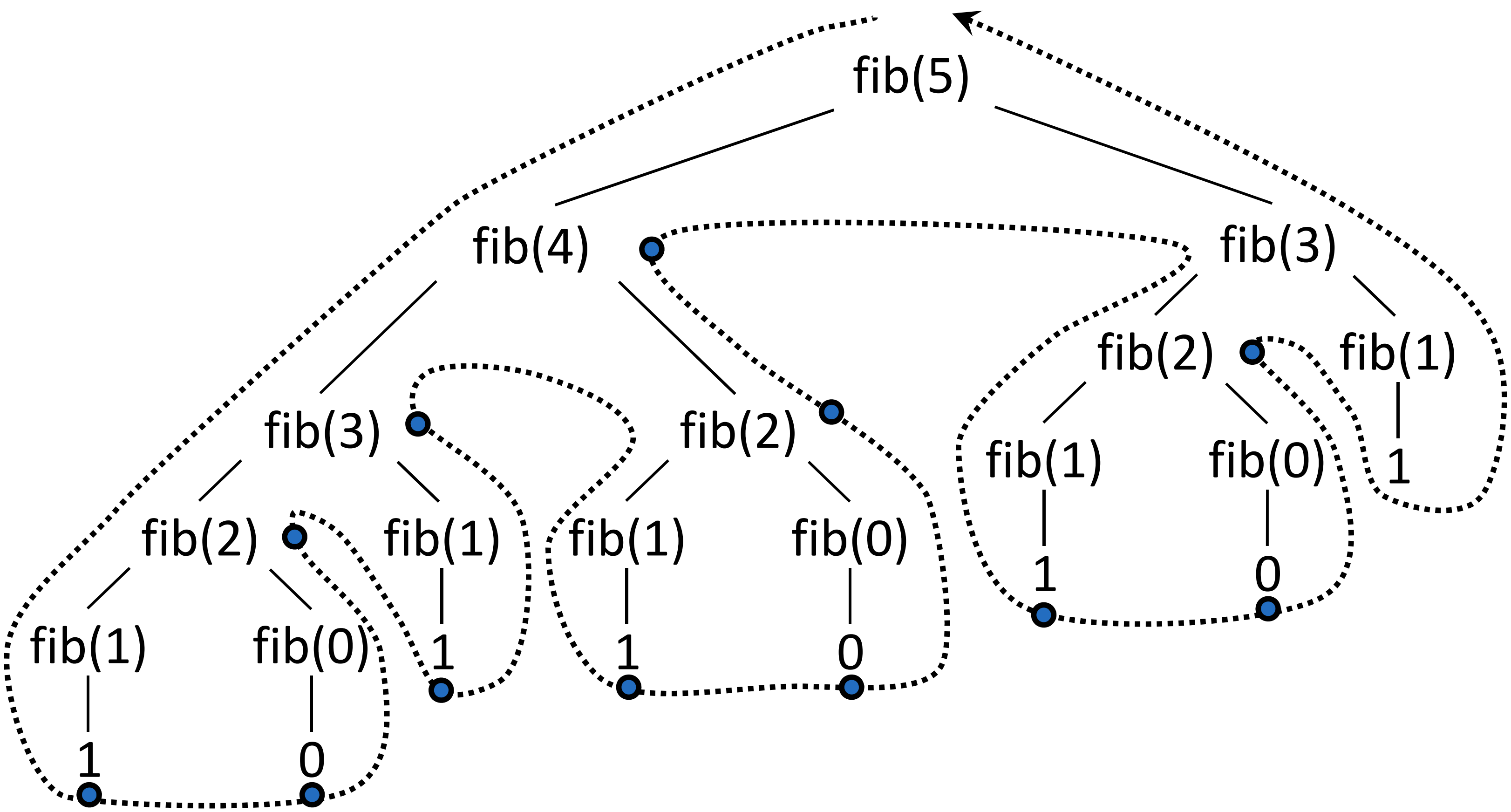

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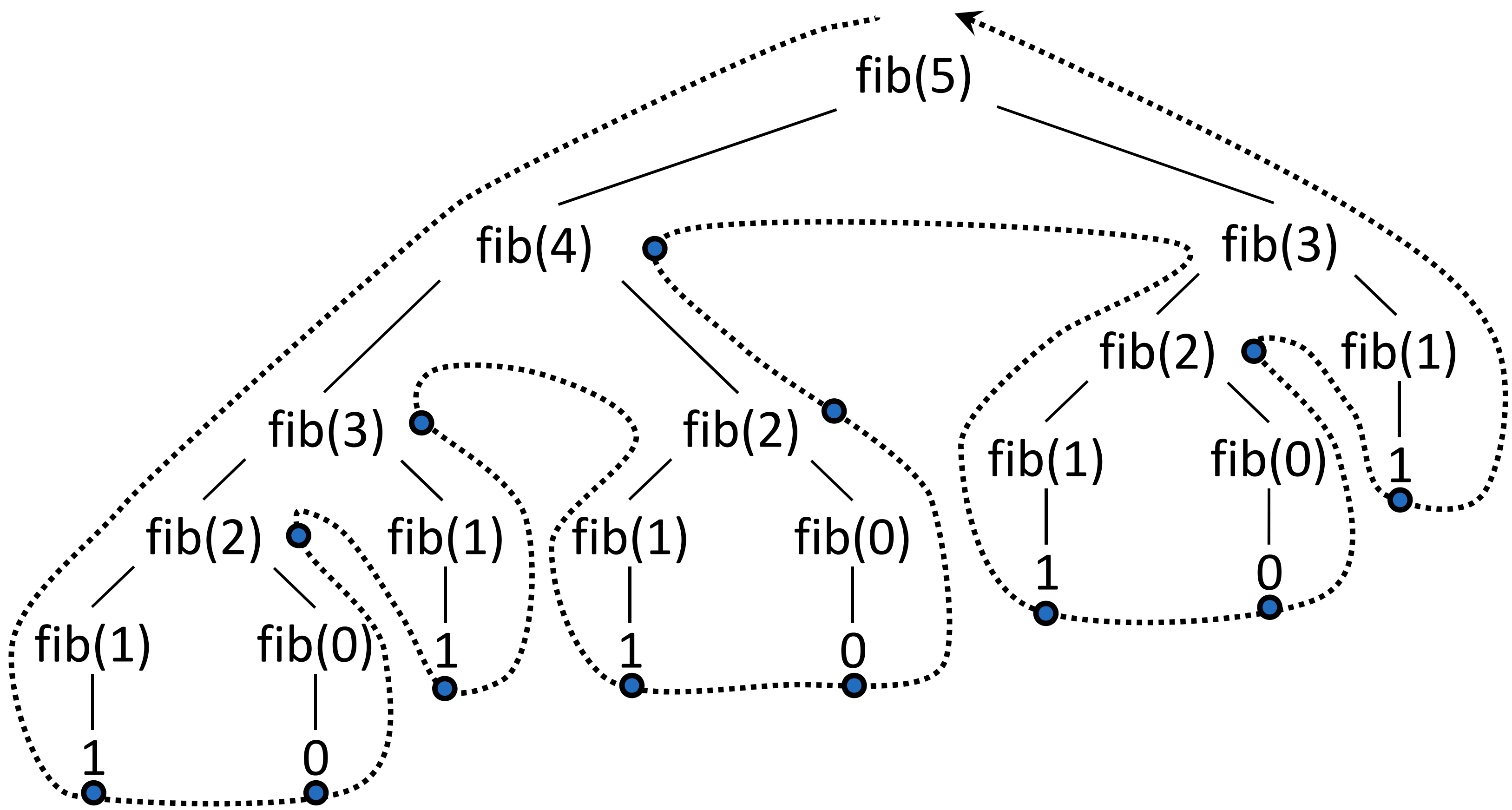

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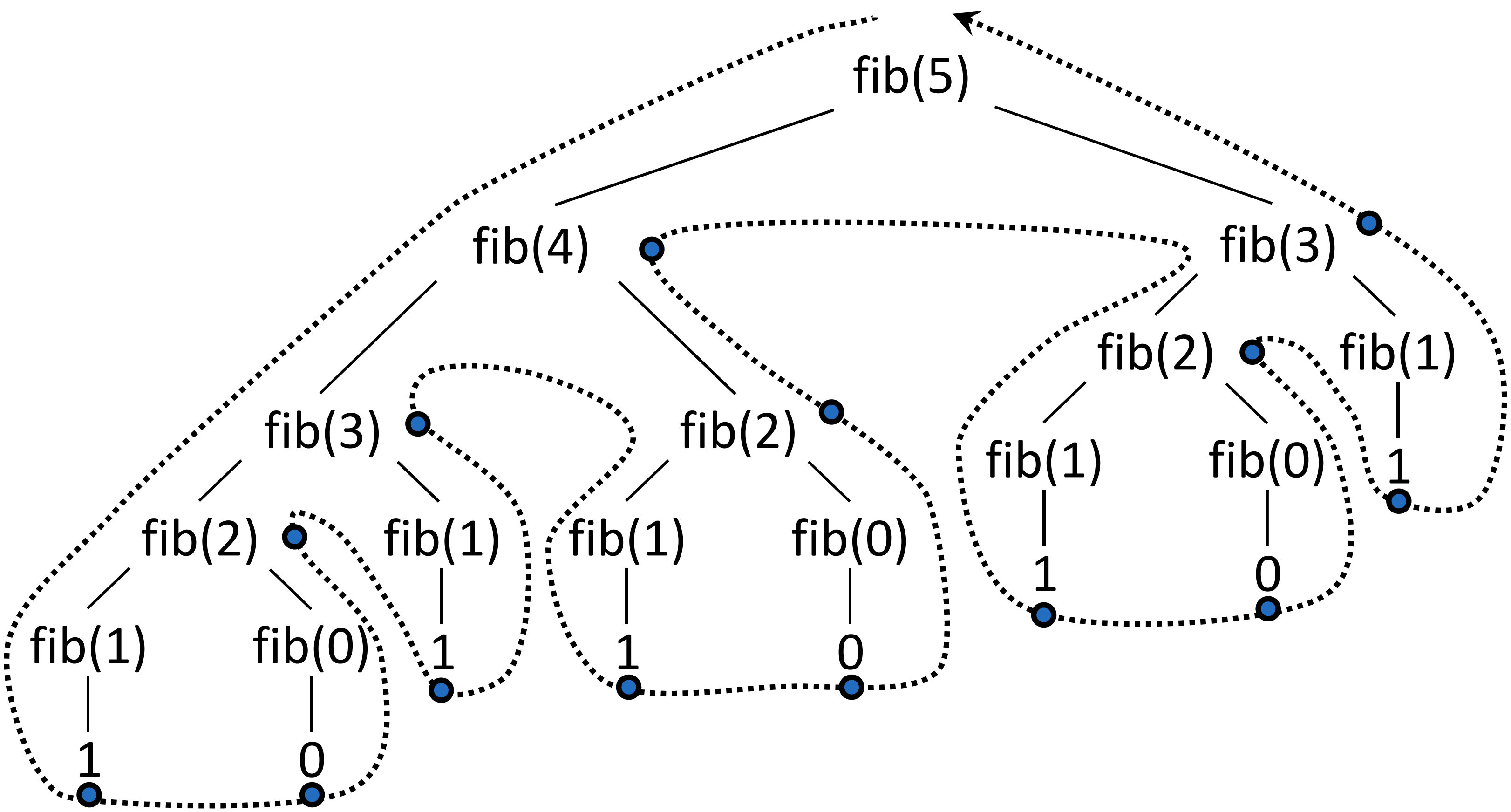

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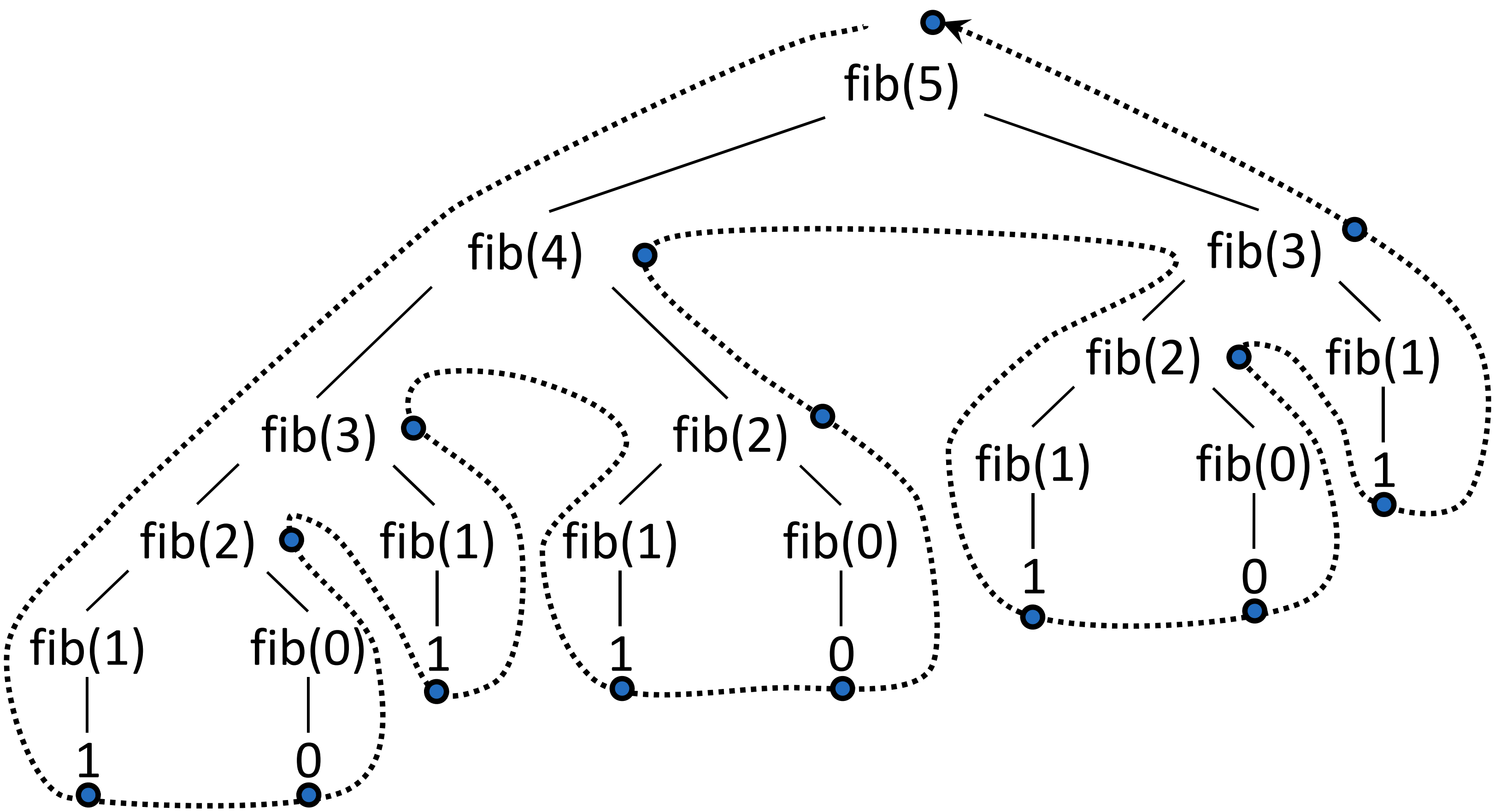

Tracing the Order of Calls

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We can use a higher-order function to see the order in which calls are made and complete

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We can use a higher-order function to see the order in which calls are made and complete

```
def trace1(fn):
    """Return a function equivalent to fn that
    also prints trace output."""
    def traced(x):
        print('Calling', fn, '(', x, ')')
        res = fn(x)
        print('Got', res, 'from', fn, '(', x, ')')
        return res
    return traced
```

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        return res
    return traced

# Rebind the name fib to a traced version of fib
fib = trace1(fib)
```

Function Decorators

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```
@trace1
def triple(x):
    return 3 * x
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def triple(x):
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Why not just use this?

```
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triple = trace1(triple)
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The Recursive Leap of Faith

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```
def factorial(n):
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Is factorial implemented correctly?

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Is factorial implemented correctly?

1. Verify the base case.

The Recursive Leap of Faith

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4. Verify that **`factorial(n)`** is correct, assuming that **`factorial(n-1)`** is correct

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```
def factorial(n):
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Oops!

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Simpler problem

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Simplifying a Problem

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Pig Latinization:

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```
def pig_latin(w):
    if starts_with_a_vowel(w):
        return w + 'ay'
    return pig_latin(rest(w) + first(w))
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Base case

Counting Change

Counting Change

\$1 = \$0.50 + \$0.25 + \$0.10 + \$0.10 + \$0.05

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\$1 = \$0.50 + \$0.25 + \$0.10 + \$0.10 + \$0.05

\$1 = 1 half dollar, 1 quarter, 2 dimes, 1 nickel

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\$1 = \$0.50 + \$0.25 + \$0.10 + \$0.10 + \$0.05

\$1 = 1 half dollar, 1 quarter, 2 dimes, 1 nickel

\$1 = 2 quarters, 2 dimes, 30 pennies

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$1 = $0.50 + $0.25 + $0.10 + $0.10 + $0.05

$1 = 1 half dollar, 1 quarter, 2 dimes, 1 nickel

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How many ways are there to change a dollar?

Counting Change

$1 = $0.50 + $0.25 + $0.10 + $0.10 + $0.05

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How many ways are there to change a dollar?

How many ways to change $0.11?

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How many ways are there to change a dollar?

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Use a dime

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10 1

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Use a dime	**No dimes**
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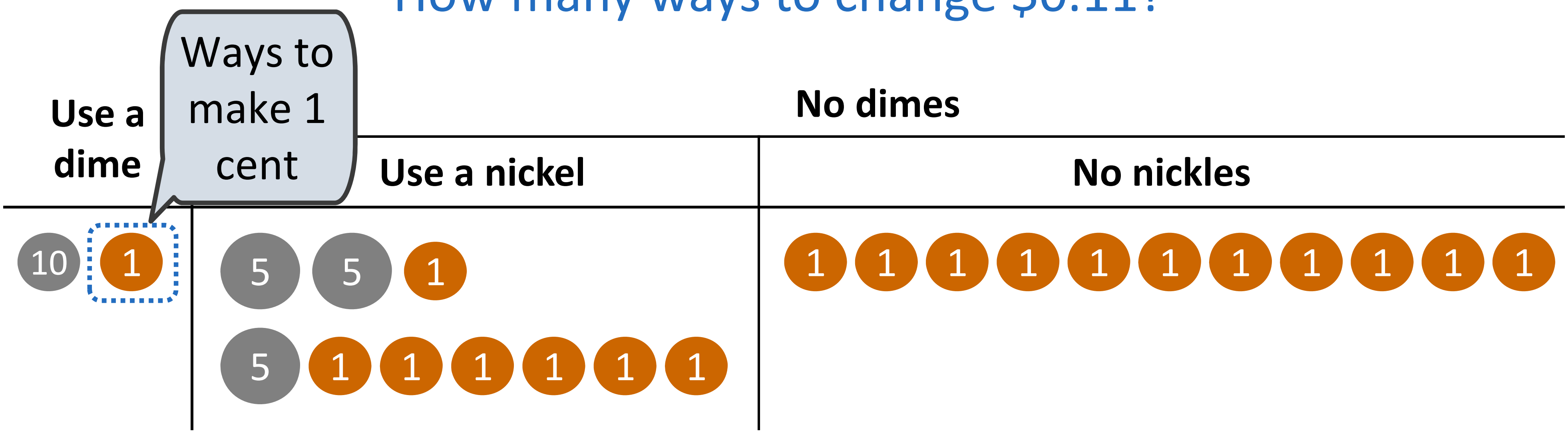

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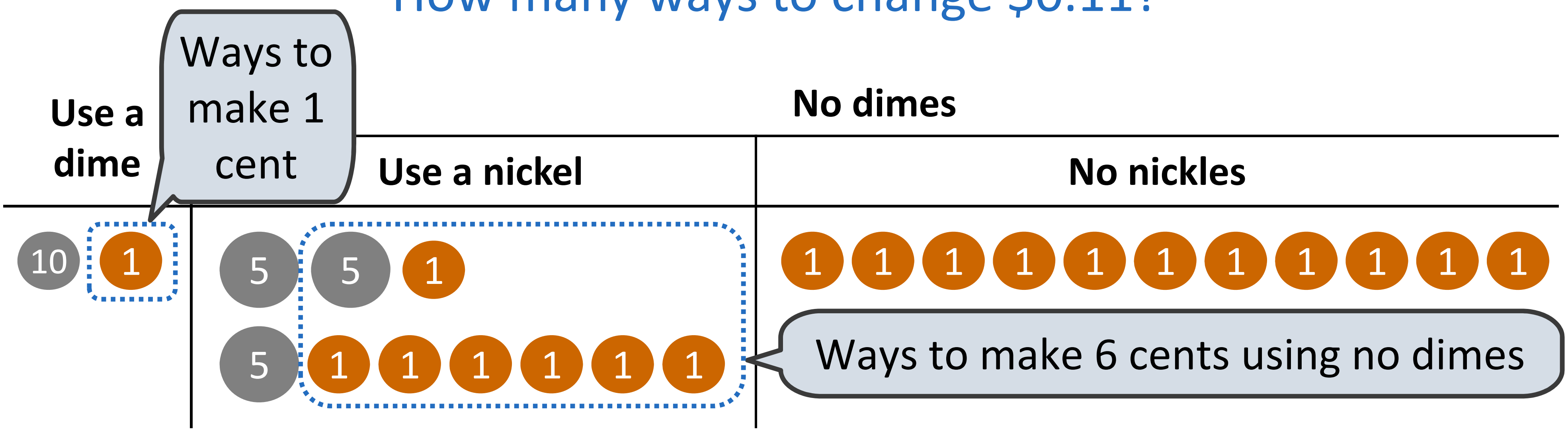
How many ways are there to change a dollar?

How many ways to change \$0.11?



Counting Change Recursively

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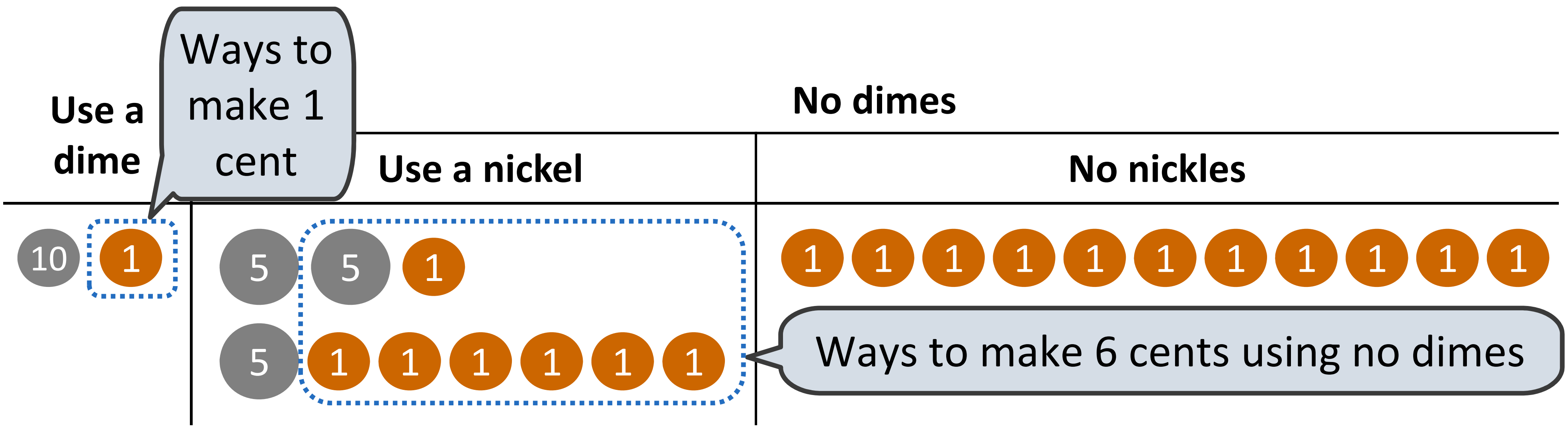

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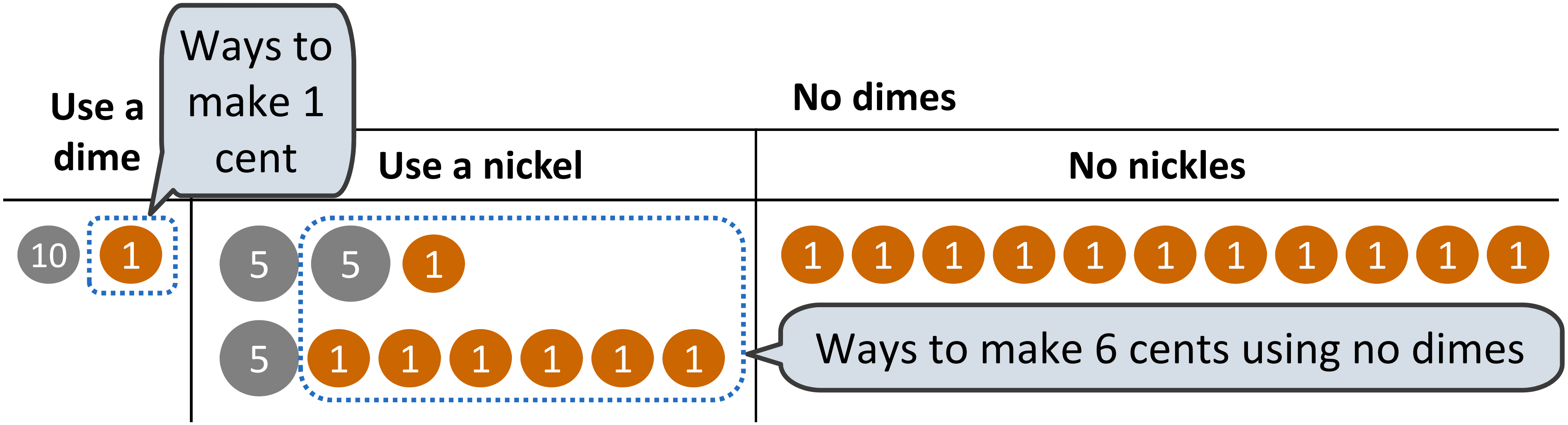
The number of ways to change an amount *a* using *n* kinds of coins is:



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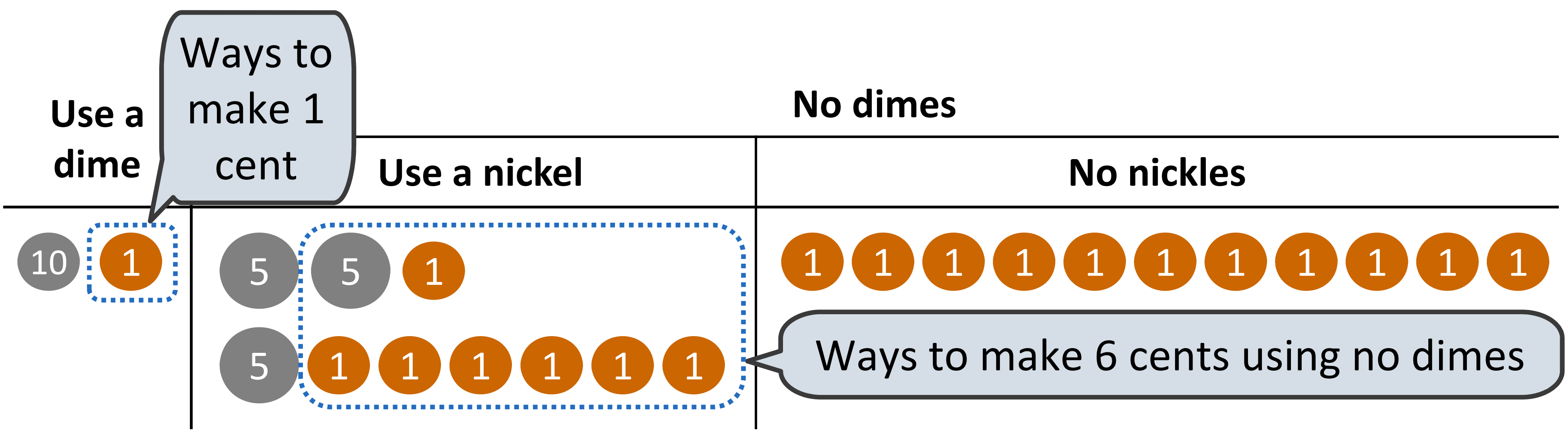
The number of ways to change an amount *a* using *n* kinds of coins is:

1. The number of ways to change *a-d* using all kinds, where *d* is the amount of the first kind of coin



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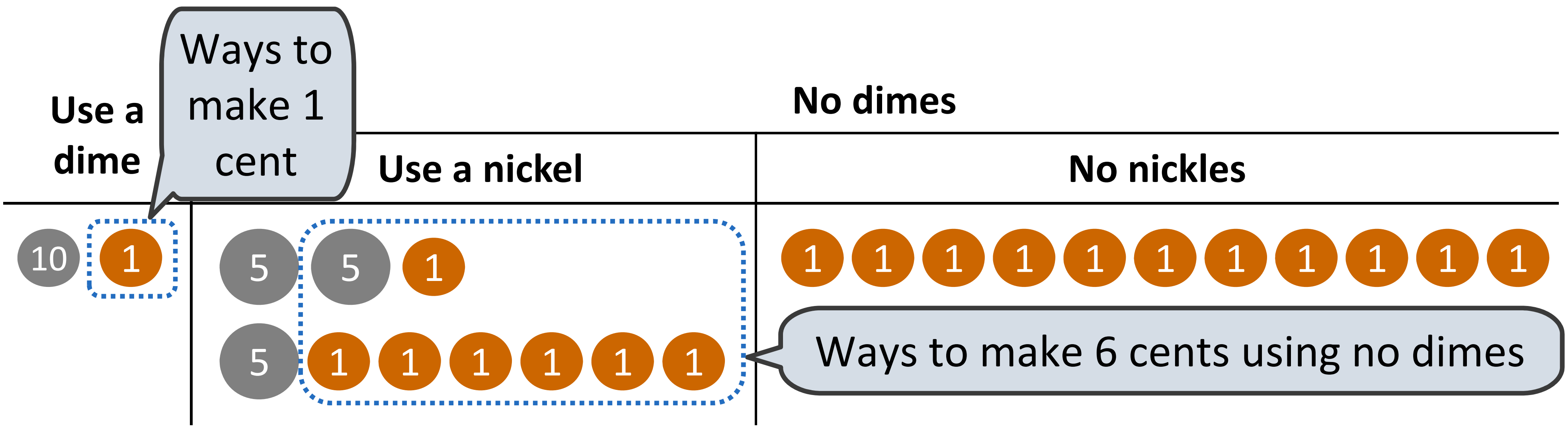
2. The number of ways to change *a* using all but the first kind



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def count_change(a, d):
    if a == 0:
        return 1
    if a < 0 or d == 0:
        return 0
    return (count_change(a-d, d) +
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Can't make negative amount, or any amount with no coins

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Functional abstraction to get next coin