CS70, Fall 2003 **Discussion #8** **Amir Kamil**
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Topics: Counting, Probability

1 Poker

We start off our discussion of probability by computing the probability of getting various hands in the game of poker[1]. A *hand* consists of five cards, which are scored according to type. (There are other rules to break ties, but we won't consider them here.) The possible types of hands are a single pair, two pairs, a triple, a full house, four of a kind, a straight, a flush, a straight flush, and a royal flush. Note that a deck of cards contains 52 cards. There are thirteen different values (the values 2 to 10, the jack, the queen, the king, and the ace), and four *suits* (diamonds, clubs, hearts, and spades), and each card has a value and a suit. As a result, there are $\binom{52}{5} = 2598960$ possible distinct poker hands. For the purposes of this discussion, we assume that five cards are randomly dealt to each player, and the player's dealt hand is final.

1. A *single pair* is a hand that contains four distinct values, one of which is repeated. Thus the hand has the form $AABCD$, where each letter corresponds to a distinct value . In order to create such a hand, we can first choose the value of A, then the actual cards we are using that have value A (for example, if our value is 2, we have a choice of the 2 of clubs, the 2 of hearts, the 2 of diamonds, and the 2 of spades), then the values of B, C, and D, and finally the actual cards of value B, C, and D. Now we have 13 choices for A, $\binom{4}{2}$ choices for the cards of value A, $\binom{12}{3}$ choices for B, C, and D (we can't pick the same value as A), and 4 choices each for the cards of value B, C, and D. Thus the number of hands that are single pair is $13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240$. Thus the probability of getting a single pair is $\frac{1098240}{2598960} = 0.423$.

   Note that the number of single pairs is *not* $13 \cdot \binom{4}{2} \cdot \binom{12}{1} \cdot \binom{12}{1} \cdot \binom{12}{1} \cdot 4^3$, which corresponds to choosing B after choosing A and its cards, then choosing C, and then choosing D. This is because permuting our choices for B, C, and D does not change a hand (e.g. picking $B = 2$, $C = 3$, $D = 4$, is equivalent to picking $B = 4$, $C = 3$, $D = 2$), so we have to divide by the number of permutations of B, C, and D, or 3!, recovering our previous result.

2. A *two pair* is a hand of the form $AABBC$, where each letter is a distinct value. We first choose the values of A and B (remember that if we choose first one, then the other, we have to divide by 2!), then the value of C. Furthermore, we have to choose two cards of value A, two of value B, and one of value C. Thus the number of two pair hands is $\binom{13}{2} \cdot \binom{11}{1} \cdot \binom{4}{2}^2 \cdot 4 = 123552$, and the probability of getting such a hand is 0.0475.

3. A *triple* has the form $AAABC$. First choosing the value of A, then of B and C, and then choosing the cards of value A, B, and C gives us $13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4^2 = 54912$ different hands, for a probability of 0.0211.

   Counting the number in a different way, we can first choose the value of A, then of B, then of C, and then the cards of each value. But then we must divide by 2! for permuting B and C, so the result is $13 \cdot 12 \cdot 11 \cdot \binom{4}{3} \cdot 4^2 \cdot \frac{1}{2!} = 54912$ hands, the same as before.

4. A *full house* is a hand of the form $AAABB$. There are $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 3744$ such hands, and the probability of getting such a hand is 0.00144.

5. A hand of the form $AAAAB$ is a *four of a kind*. The number of four of a kinds is $13 \cdot 12 \cdot \binom{4}{4} \cdot 4 = 624$, and the probability of getting one is a measly 0.000240.

[1]The material in this section was adapted from notes by Tom Ramsey of the University of Hawaii.

6. A *straight* is a set of cards in order, $\{4,5,6,7,8\}$ for example. The ace is allowed to be either 1 or 14, so $\{A,2,3,4,5\}$ and $\{10,J,Q,K,A\}$ are straights, but $\{J,Q,K,A,2\}$ is not. Note that choosing the low value in a straight, for which there are 10 choices, forces the values of the other cards, and then all that is left is to choose the suits of each card. Thus the number of straights is $13 \cdot 4^5 = 10240$.

   However, we have overcounted slightly, since we included straight and royal flushes. From below, there are 36 straight flushes and 4 royal flushes, so the number of straights is actually 10200. The probability of getting a straight is 0.00392.

7. A *flush* is a hand in which all the cards are of the same suit. There are four choices for the suit, and then five values must be chosen out of 13, so the number of flushes is $4 \cdot \binom{13}{5} = 5148$.

   Again, we have to subtract the number of straight and royal flushes, so there are actually 5108 straights. The corresponding probability is 0.00197.

8. A *straight flush* is the intersection of a straight and a flush. (Well, it would be, if we hadn't removed straight flushes in defining a straight and a flush.) There are four suits and 10 possible low cards, so there are 40 straight flushes.

   But once again we counted royal flushes as well. Subtracting the number of royal flushes, we get 36 possible straight flushes. The probability of getting one is only 0.0000139.

9. Finally, the mother of all hands is the *royal flush*. These are the hands consisting of a ten, jack, queen, king, and ace, all of the same suit. There are only 4 of these, and an inconsequential probability of getting one, 0.00000154.

10. The last type of hand is one that stinks. It has the form $ABCDE$. There are $\binom{13}{5} - 10$ choices for the values (subtracting the choices that result in straights). There are $4^5 - 4$ choices for the actual cards (subtracting the choices that result in flushes). Thus the number of hands that stink is $(\binom{13}{5} - 10) \cdot (4^5 - 4) = 1302540$. The probability of stinking is 0.501.

Of course, knowing what an opponent has changes the probability of getting each hand (the probability is now conditioned on the opponent's hand). For example, if you are playing against a single opponent who you know has a royal flush, the probability of you having one as well decreases to $\frac{3}{\binom{47}{5}}$.