CS70, Fall 2003 **Discussion #2** **Amir Kamil**
UC Berkeley **9/5/03**

Topics: Administrivia, Simple Induction, Bad Proofs, Strong Induction

1 Administrivia

- Do NOT turn in homework in class. Turn it in the homework box in 283 Soda.
- If you come to office hours and I'm not there, check 566 Soda (my office).

2 Simple Induction

Consider the claim "$n^n > n!$ for all natural numbers $n > 1$." If the claim is false, then it may be easy to disprove it, since only a single counterexample needs to be found. If it is true, however, it is hard to prove so. One way to prove such a claim is by *induction*. It is necessary only to prove that the claim is true for the smallest number in the set of numbers for which it claims to be true, and that if it is true for one number in the set, it is true for the next number.

So let's actually get around to proving the claim. The first thing that must be done in an inductive proof is to write a proposition that depends on an input. For example, we can write

$$P(n) = n^n > n!.$$

Then our claim in terms of the proposition is

$$\forall n \in \mathbf{N}.\ n > 1 \Longrightarrow P(n).$$

Now all we have to do is prove that $P(n)$ is true for the 2 (the *base case*), and if it is true for n, it is true for $n+1$ (the *inductive step*). (Note that there is a small technicality we are ignoring. The previous two steps only prove that the proposition $n > 1 \Longrightarrow P(n)$ is true if the antecedent is true. However, since an implication is always true when the antecedent is false, we can ignore that case.)

Theorem 2.1: $\forall n \in \mathbf{N}.\ n > 1 \Longrightarrow P(n)$
Proof by induction:

- Base case: $n = 2$
  $2^2 > 2!$, i.e. $4 > 2$ is true.
- Inductive step:
  We need to show that $P(n) \Longrightarrow P(n+1)$, or $n^n > n! \Longrightarrow (n+1)^{n+1} > (n+1)!$. By the inductive hypothesis, $n^n > n!$. Then
  1. $n^n > n!$
  2. $(n+1)n^n > (n+1)n!$ multiplying by $n+1$, since $n+1 > 0$
  3. $(n+1)^n > n^n$ since $a > b \Longrightarrow a^c > b^c$ when a, b, and c are positive
  4. $(n+1)^{n+1} > (n+1)n^n$ multiplying by $n+1$, since $n+1 > 0$
  5. $(n+1)^{n+1} > (n+1)n!$ by transitivity of inequalities
  6. $(n+1)^{n+1} > (n+1)!$ by definition of factorial

Thus by the principle of induction, the claim is true.

Note that induction is not always the best way to approach a proof. Consider the slightly modified claim "$\frac{n}{3}^{\frac{n}{3}} < n!$ for all natural numbers $n > 0$." If you try writing an inductive proof for this claim, you will see that it is not that easy. In fact, it is much easier to write a proof by arbitrary example for this claim.

3 Bad Proofs

So we've already "proven" that $2 = 1$. This time, let's prove an even stronger claim: "all natural numbers greater than or equal to 1 are equal to 1." In terms of the proposition

$$P(n) = n = 1,$$

our claim is

$$\forall n \in \mathbf{N}.\ n \geq 1 \Longrightarrow P(n).$$

Here's the proof:

Claim: $\forall n \in \mathbf{N}.\ n \geq 1 \Longrightarrow P(n)$
Proof by induction:

- Base case: $n = 1$
  $1 = 1$ is true.
- Inductive step: $n = 1 \Longrightarrow n + 1 = 1$
  1. Suppose $n + 1 = 1$.
  2. Then $(n + 1) \cdot (n - 1) = n - 1$, since $a = b \Longrightarrow a \cdot c = b \cdot c$.
  3. Then $n^2 - 1 = n - 1$, by the distributive property.
  4. Then $n^2 = n$, since $a = b \Longrightarrow a + c = b + c$.
  5. Then $1 = 1$, by the inductive hypothesis (substituting 1 for n).
  6. This is true, so $n = 1 \Longrightarrow n + 1 = 1$.

Therefore, by the principle of induction, $\forall n \in \mathbf{N}.\ n \geq 1 \Longrightarrow P(n)$.

Is anything wrong with the above proof? (Of course, since otherwise it wouldn't be in the section entitled "Bad proofs.") Can you find the bug in the proof?

The error in the proof is an example of a *converse error* for proofs. In the inductive step, it starts with the conclusion and derives the premise. This is similar to using $Q \Longrightarrow P$ to try to prove that $P \Longrightarrow Q$. It is not valid to do so.

In some cases, the steps in such a proof can be reordered to result in a valid proof. In this case, however, reordering the steps results in division by zero in the place of the multiplication by zero in step 2, invalidating the proof.

There are multiple examples of bad inductive proofs. Some are just stylistically bad, such as a proof that is composed completely of symbols. Each step in a proof should be part of an English sentence, and should be justified rigorously. Examples of real fallacies in inductive proofs include the converse error above, ignoring or otherwise making a mistake in the base case, and the same logical errors that affect non-inductive proofs. Be careful not to fall for any of these in your proofs.

4 Strong Induction

Consider the following claim: "all natural numbers greater than 1 can be expressed as the sum of primes." Let's attempt to prove this claim inductively. First, define the proposition

$$P(n) = \text{n can be expressed as the sum of primes.}$$

Then the above claim is

$$\forall x \in \mathbf{N}.\ x > 1 \Longrightarrow P(x).$$

Now unfortunately, 1 is not a prime number, so $P(x)$ does not immediately imply $P(x+1)$. However, 2 is a prime number, so $P(x)$ obviously implies $P(x+2)$. So let's consider a new proposition

$$Q(n) = P(n) \wedge P(n-1)$$

Then the above claim is

$$\forall x \in \mathbf{N}.\ x > 2 \Longrightarrow Q(x).$$

Now we can prove the claim using induction.

Theorem 2.2: $\forall x \in \mathbf{N}.\ x > 2 \Longrightarrow Q(x)$.
Proof by induction:

- Base cases: $Q(3) = P(2) \wedge P(3)$
  Trivially true, since 2 and 3 are prime, and the can be expressed as the sum of themselves.
- Inductive step:
  We need to show that $Q(n) \Longrightarrow Q(n+1)$, or $P(n-1) \wedge P(n) \Longrightarrow P(n) \wedge P(n+1)$ by substition. $P(n)$ is true from applying and-elimination to the inductive hypothesis $Q(n) = P(n-1) \wedge P(n)$. Now $n+1$ can be expressed as $(n-1)+2$. By the inductive hypothesis, $n-1$ can be expressed as a sum of some k primes $p_1 + \cdots + p_k$. As such, $n+1$ can be expressed as the sum of the $k+1$ primes $p_1 + \cdots + p_k + 2$. Thus $P(n+1)$. Since $P(n)$ and $P(n+1)$, by and-introduction, $Q(n+1)$.

Since the base case and the inductive step are true, by the principle of induction, we have proven the claim.

This technique can be generalized by not only adding $P(n-1)$ to the proposition $Q(n)$, but also $P(n-2)$, $P(n-3)$, $\cdots$ up to the base case $P(2)$. Then we have to prove the implication that $P(2) \wedge P(3) \wedge \cdots \wedge P(n) \Longrightarrow P(2) \wedge P(3) \wedge \cdots \wedge P(n) \wedge P(n+1)$. But since we can apply and-elimination to prove all but $P(n+1)$ in the consequent, we only need to prove the simpler implication that $P(2) \wedge P(3) \wedge \cdots \wedge P(n) \Longrightarrow P(n+1)$. This generalization is called *strong induction* over $P(n)$. As you can see above, strong induction over $P(n)$ is equivalent to simple induction over $Q(n)$.