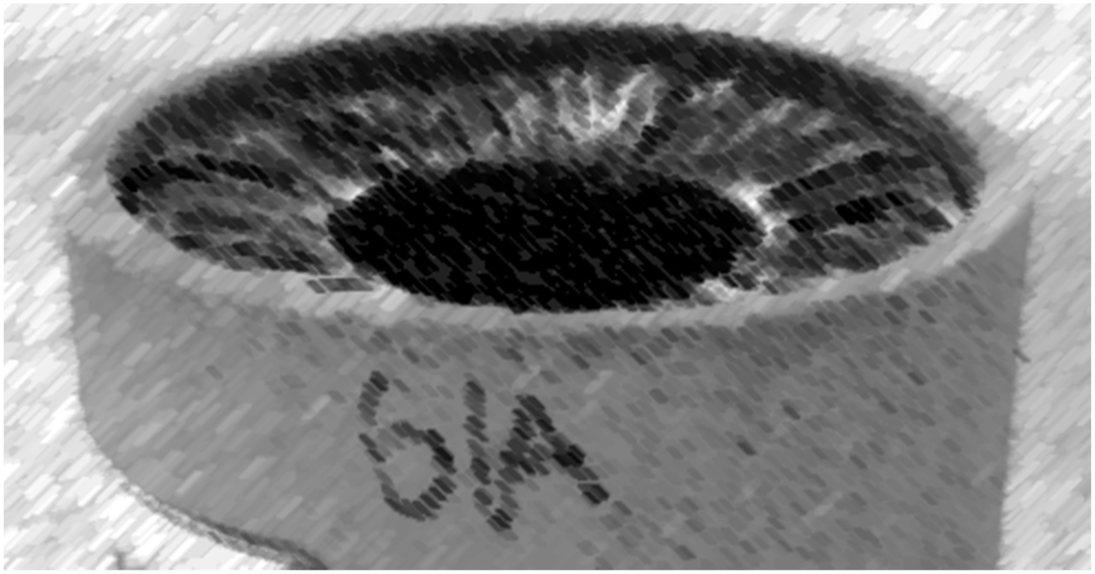

CS61A Lecture 24

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UC Berkeley

March 18, 2013

Announcements

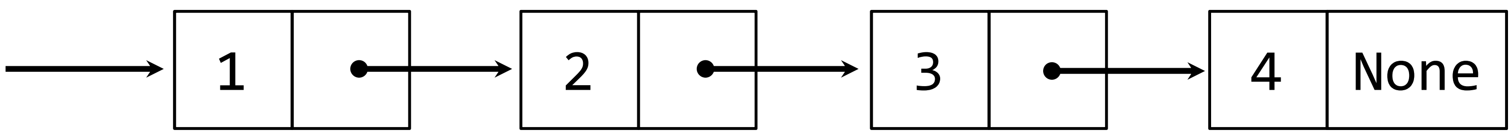
- Ants project due tonight
- HW8 due Wednesday at 7pm
- Midterm 2 Thursday at 7pm
  - See course website for more information

Closure Property of Data

A tuple can contain another tuple as an element.

Pairs are sufficient to represent sequences.

Recursive list representation of the sequence 1, 2, 3, 4:



Recursive lists are recursive: the rest of the list is a list.

Nested pairs (old): `(1, (2, (3, (4, None))))`

Rlist class (new): `Rlist(1, Rlist(2, Rlist(3, Rlist(4))))`

Recursive List Class

Methods can be recursive as well!

```
class Rlist(object):
    class EmptyList(object):
        def __len__(self):
            return 0
    empty = EmptyList()
    def __init__(self, first, rest=empty):
        self.first = first
        self.rest = rest
    def __len__(self):
        return 1 + len(self.rest)
    def __getitem__(self, i):
        if i == 0:
            return self.first
        return self.rest[i - 1]
```

There's the base case!

Yes, this call is recursive

Recursive Operations on Rlists

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>>> s = Rlist(1, Rlist(2, Rlist(3)))

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>>> extend_rlist(s.rest, s)
```

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>>> s = Rlist(1, Rlist(2, Rlist(3)))

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Rlist(2, Rlist(3, Rlist(1, Rlist(2, Rlist(3)))))
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```
def extend_rlist(s1, s2):
```

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Rlist(2, Rlist(3, Rlist(1, Rlist(2, Rlist(3)))))
```

```
def extend_rlist(s1, s2):
    if s1 is Rlist.empty:
```

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```

```
def extend_rlist(s1, s2):
    if s1 is Rlist.empty:
        return s2
    return Rlist(s1.first, extend_rlist(s1.rest, s2))
```

Map and Filter on Rlists

Map and Filter on Rlists

We want operations on a whole list, not an element at a time.

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def map_rlist(s, fn):
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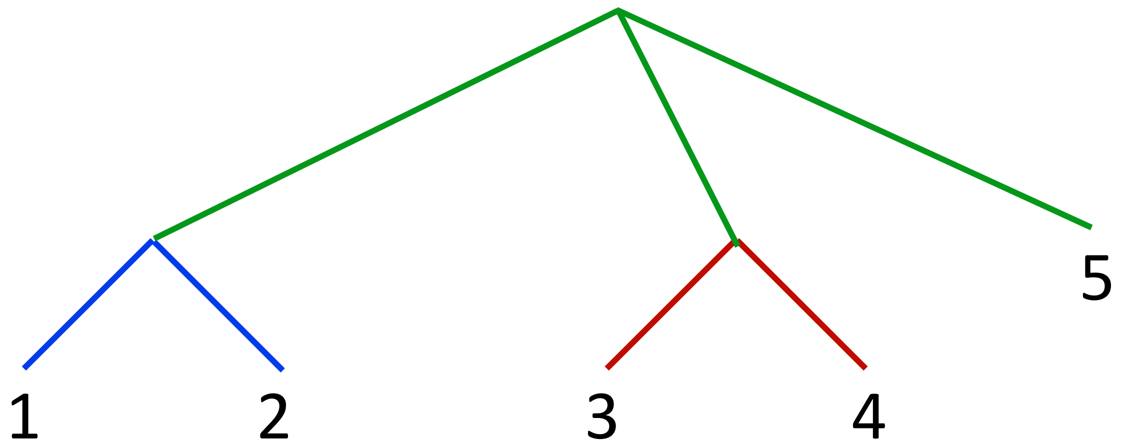

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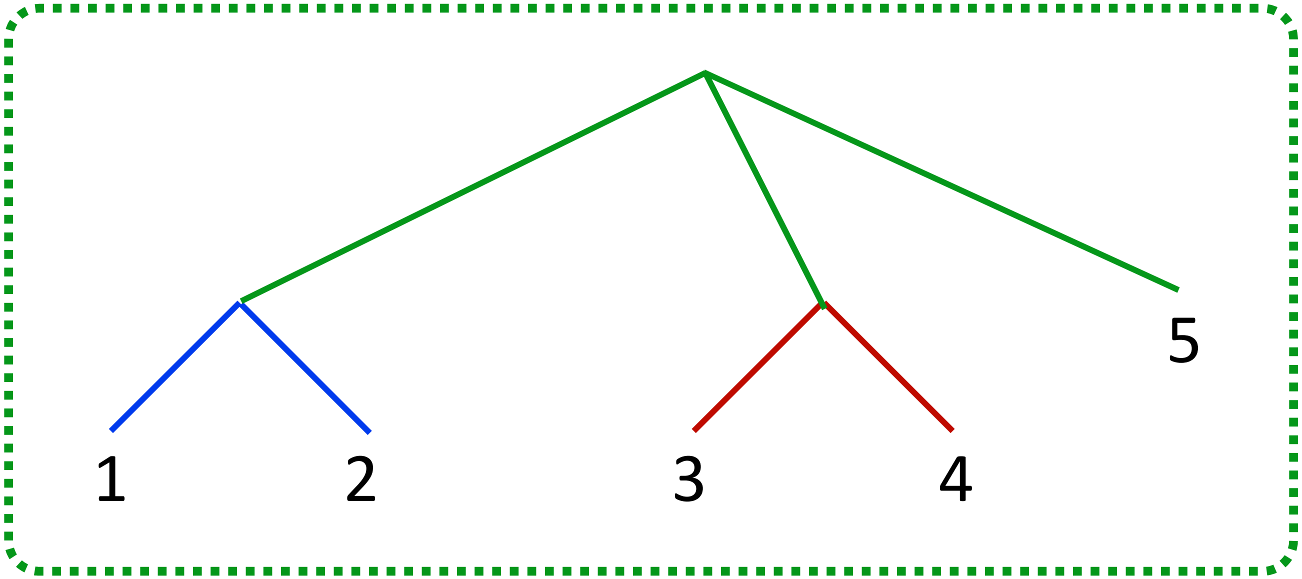

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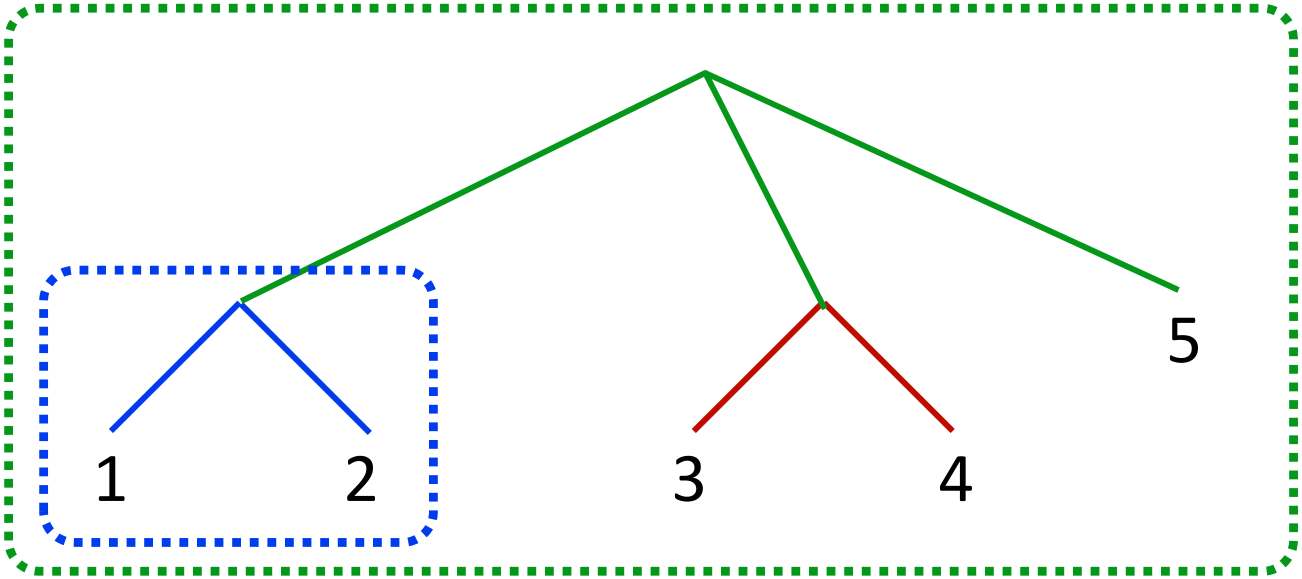

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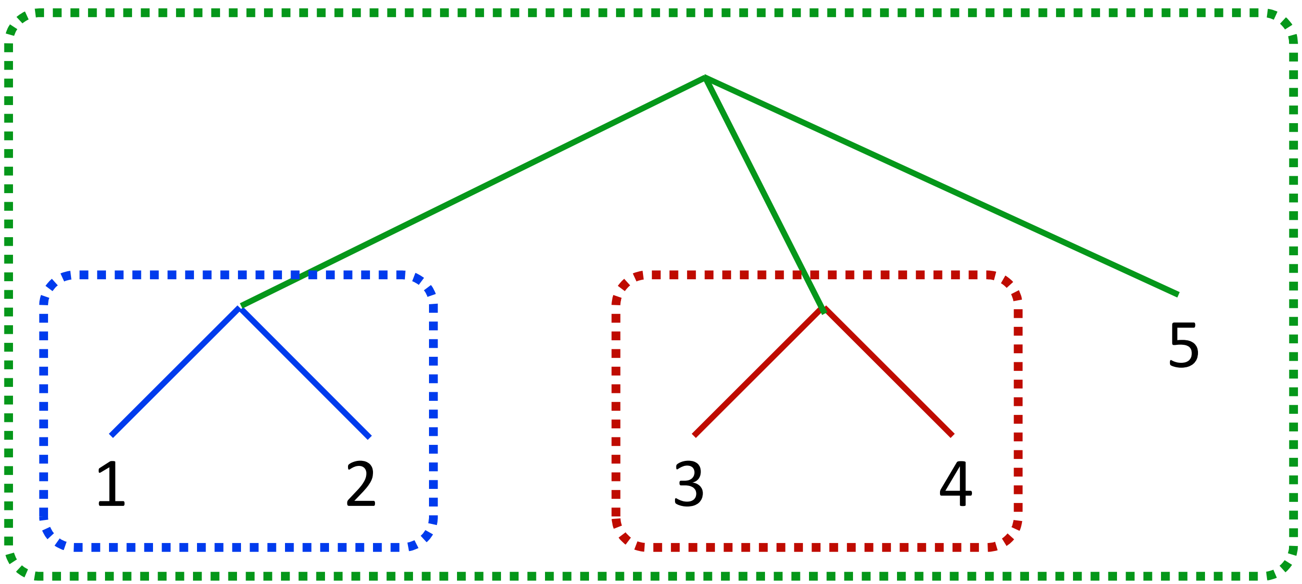

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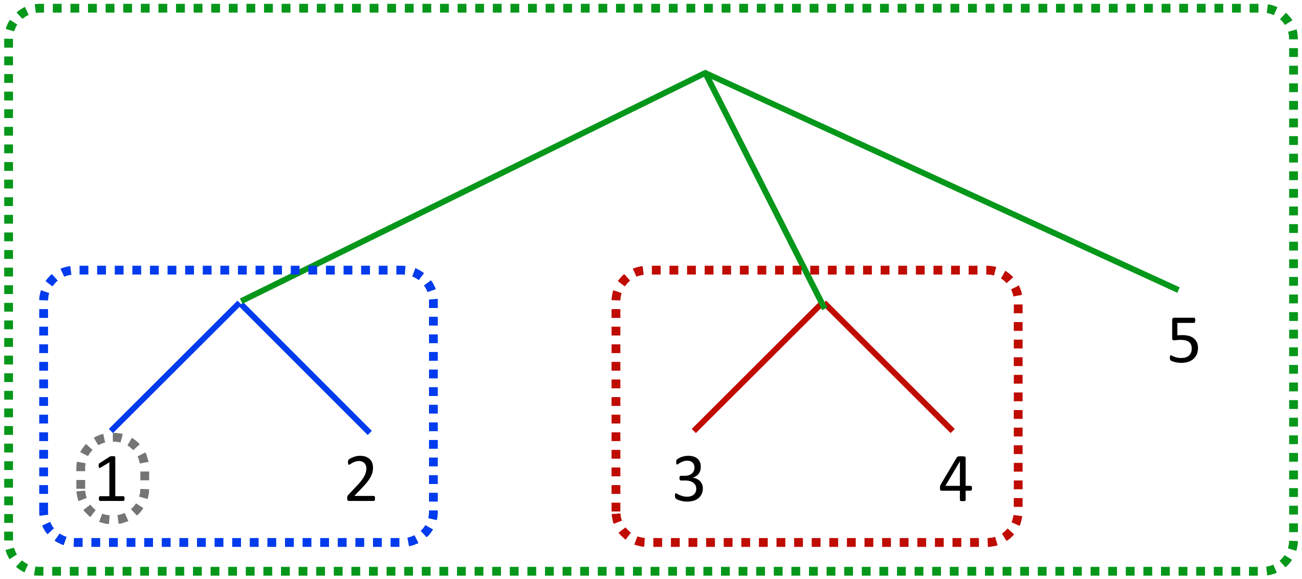

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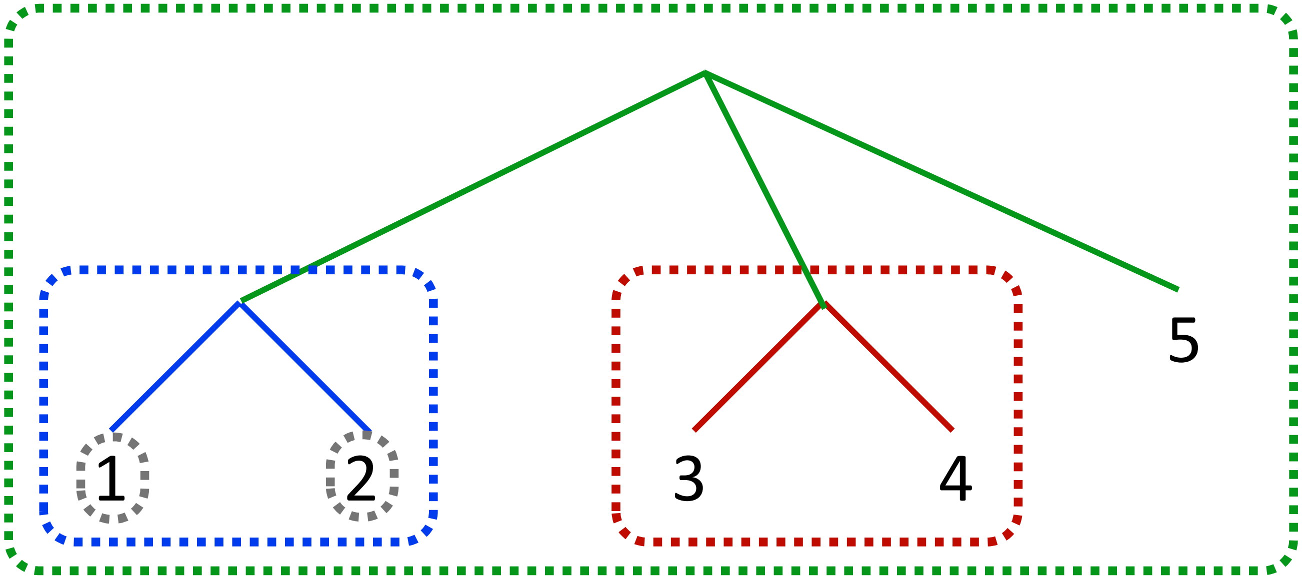

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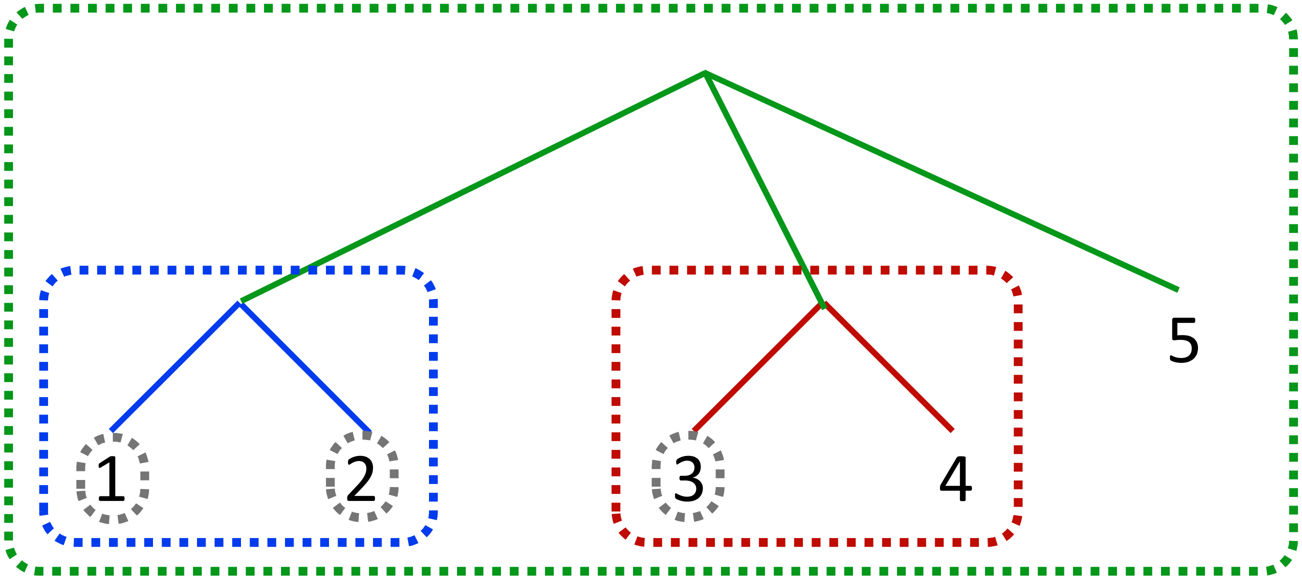

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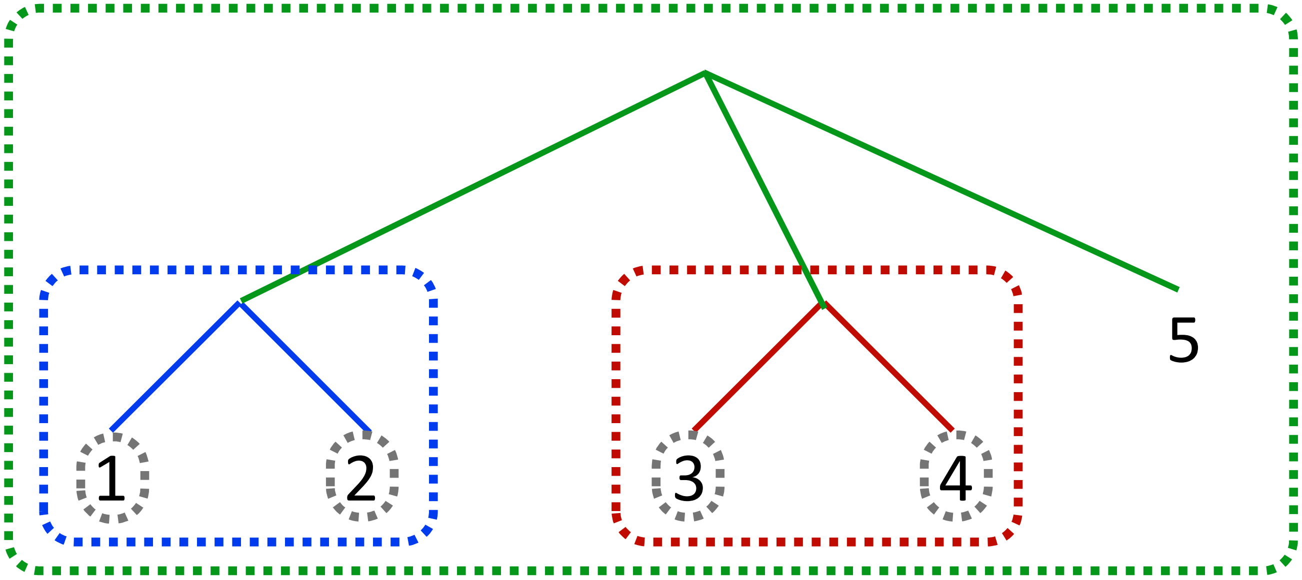

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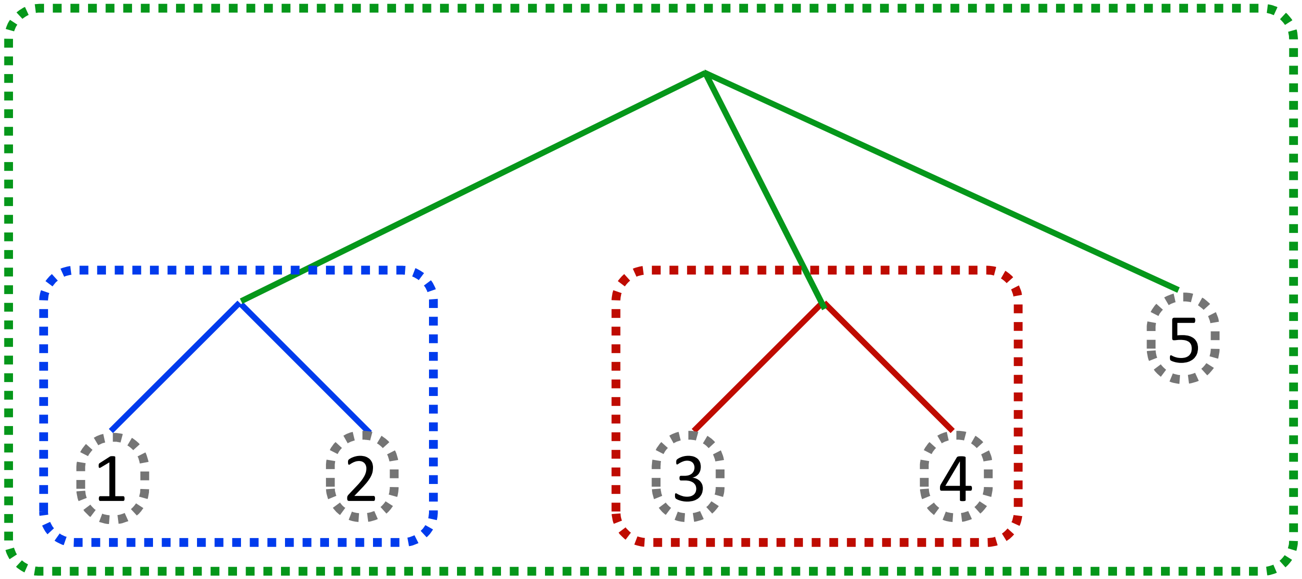

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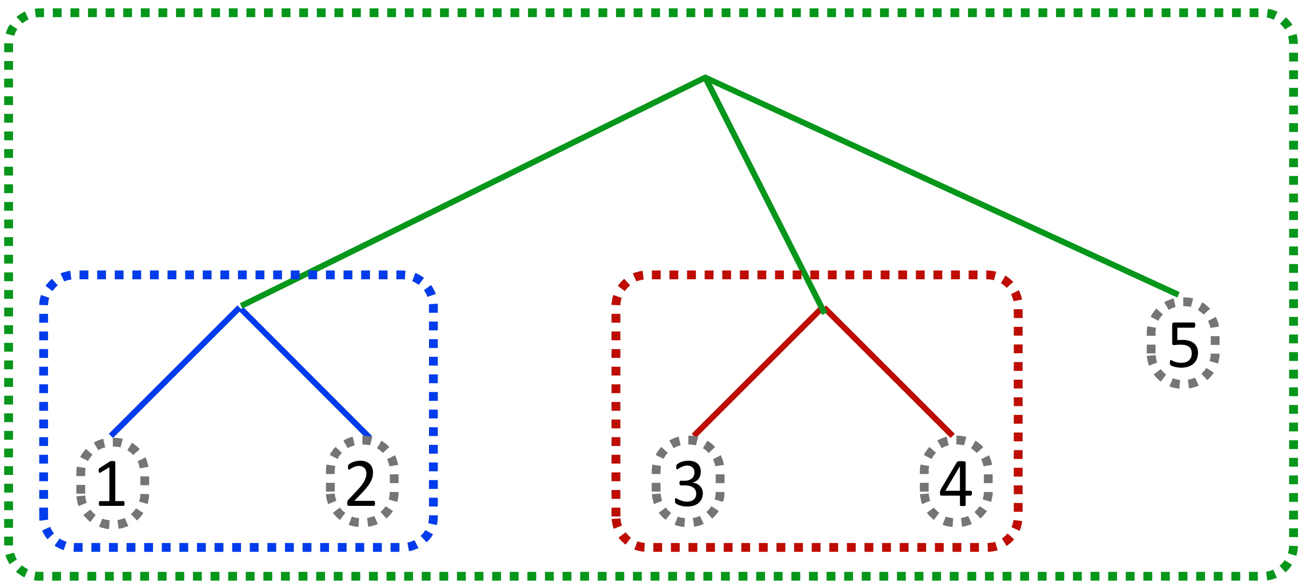

In every tree, a vast forest

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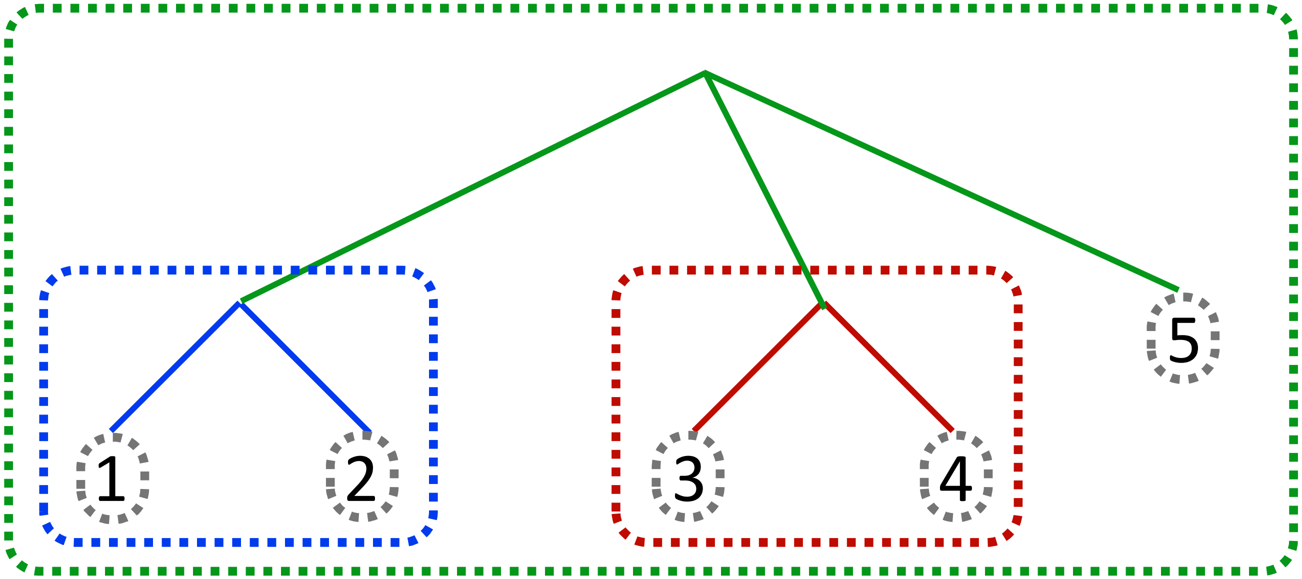

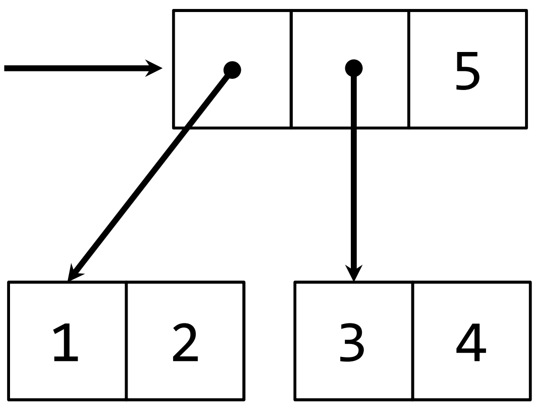

In every tree, a vast forest

Example: http://goo.gl/0h6n5

Recursive Tree Processing

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Tree operations typically make recursive calls on branches

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```
def count_leaves(tree):
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def count_leaves(tree):
    if type(tree) != tuple:
```

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def count_leaves(tree):
    if type(tree) != tuple:
        return 1
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def count_leaves(tree):
    if type(tree) != tuple:
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    return sum(map(count_leaves, tree))
```

Recursive Tree Processing

Tree operations typically make recursive calls on branches

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def count_leaves(tree):
    if type(tree) != tuple:
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def map_tree(tree, fn):
```

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```

```
def map_tree(tree, fn):
    if type(tree) != tuple:
        return fn(tree)
```

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Tree operations typically make recursive calls on branches

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def count_leaves(tree):
    if type(tree) != tuple:
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def map_tree(tree, fn):
    if type(tree) != tuple:
        return fn(tree)
    return tuple(map_tree(branch, fn)
```

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def map_tree(tree, fn):
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    return tuple(map_tree(branch, fn)
                 for branch in tree)
```

Trees with Internal Node Values

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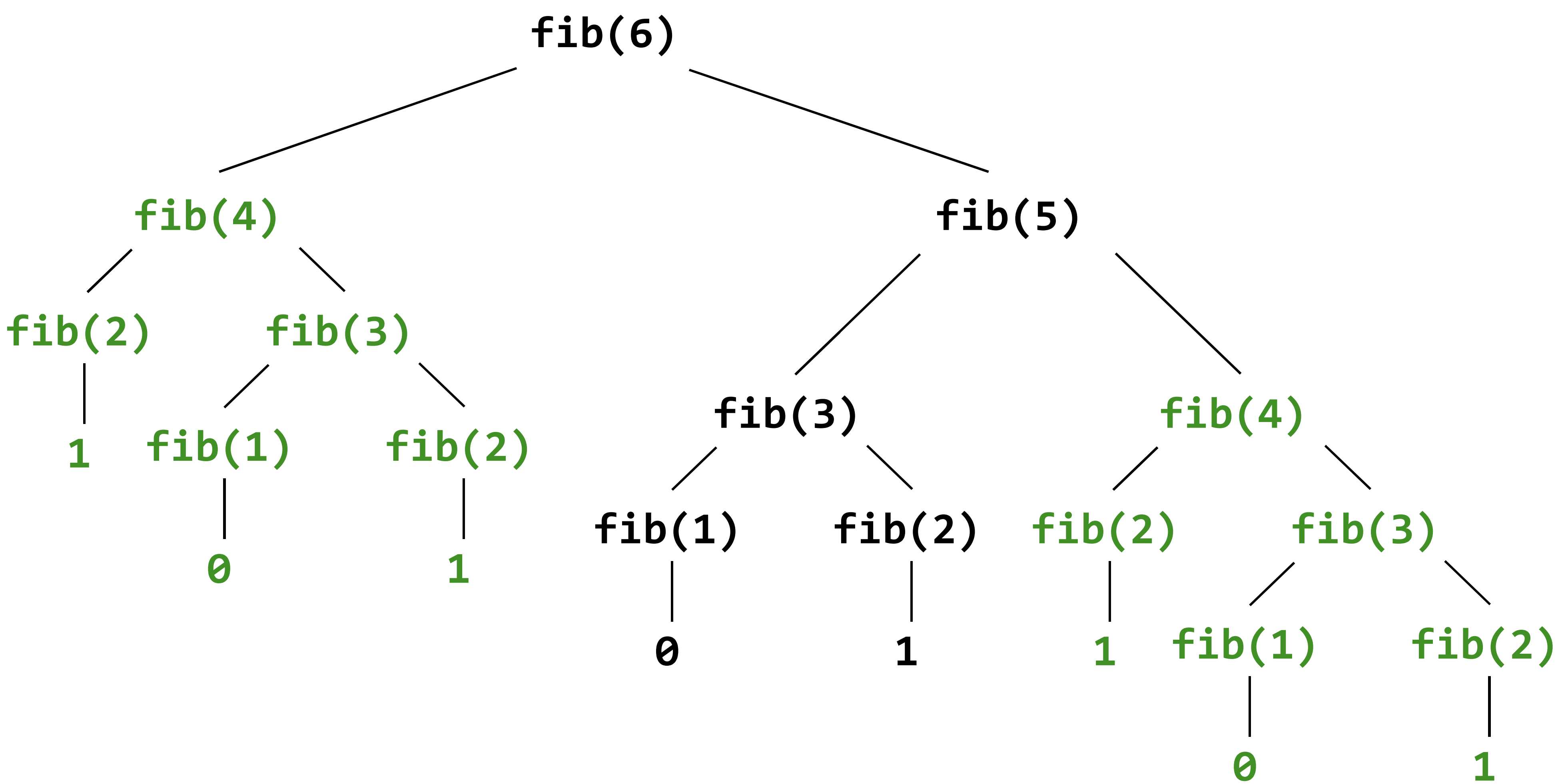

The Consumption of Time

Implementations of the same functional abstraction can require different amounts of time to compute their result.

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def count_factors(n):

    factors = 0
    for k in range(1, n + 1):
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    return factors
```

```
    sqrt_n = sqrt(n)
    k, factors = 1, 0
    while k < sqrt_n:
        if n % k == 0:
            factors += 2
        k += 1
    if k * k == n:
        factors += 1
    return factors
```

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$\lfloor \sqrt{n} \rfloor$

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for sufficiently large values of ***n***.

Constant Time: $\Theta(1)$

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def g(n):
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def foo(n):
    baz = 7
    if n > 5:
        baz += 5
    return baz
```

Constant Time: $\Theta(1)$

Time does **not** depend on input size.

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def g(n):
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def foo(n):
    baz = 7
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    return baz

def is_even(n):
    return n % 2 == 0
```

Iteration vs. Tree Recursion (Time)

Iterative and recursive implementations are not the same.

Time

```
def fib_iter(n):
    prev, curr = 1, 0
    for _ in range(n - 1):
         prev, curr = curr, prev + curr
    return curr


def fib(n):
    if n == 1:
        return 0
    if n == 2:
        return 1
    return fib(n - 2) + fib(n - 1)
```

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Next time, we will see how to make recursive version faster.

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$\Theta(\sqrt{n})$

Exponentiation

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def square(x):
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def fast_exp(b, n):
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Exponentiation

Goal: one more multiplication lets us double the problem size.

Time	Space
$\Theta(n)$	$\Theta(n)$

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```

	Time	Space
exp	$\Theta(n)$	$\Theta(n)$
fast_exp	$\Theta(\log n)$	$\Theta(\log n)$

The Consumption of Space

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Which environment frames do we need to keep during evaluation?

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Active environments:

- Environments for any statements currently being executed
- Parent environments of functions named in active environments

The Consumption of Space

Implementations of the same functional abstraction can require different amounts of time to compute their result.

```
def count_factors(n):

    factors = 0
    for k in range(1, n + 1):
        if n % k == 0:
            factors += 1
    return factors
```

```
    sqrt_n = sqrt(n)
    k, factors = 1, 0
    while k < sqrt_n:
        if n % k == 0:
            factors += 2
        k += 1
    if k * k == n:
        factors += 1
    return factors
```

Time	Space

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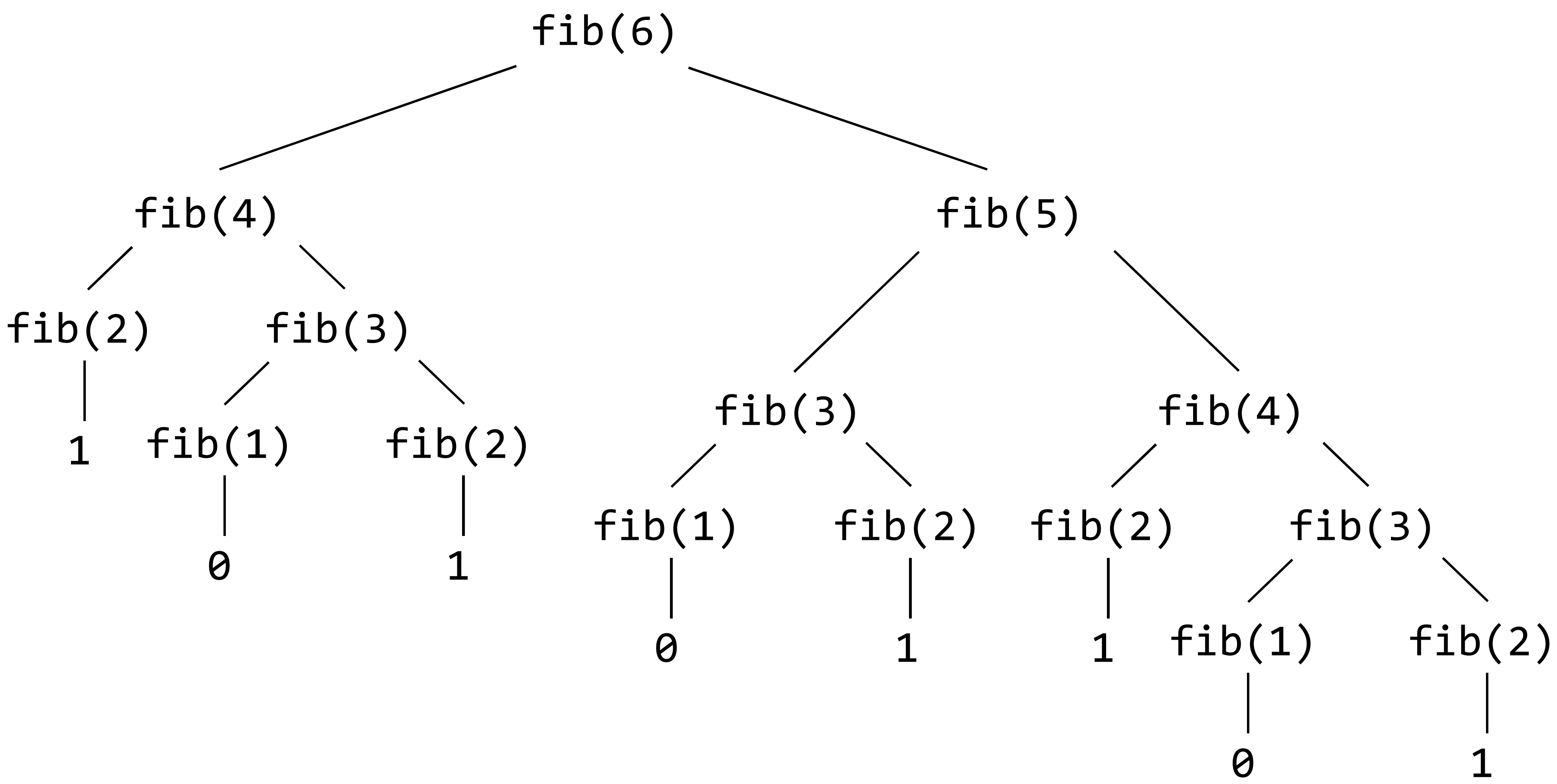
Time	Space
$\Theta(n)$	$\Theta(1)$
$\Theta(\sqrt{n})$	$\Theta(1)$

Fibonacci Memory Consumption



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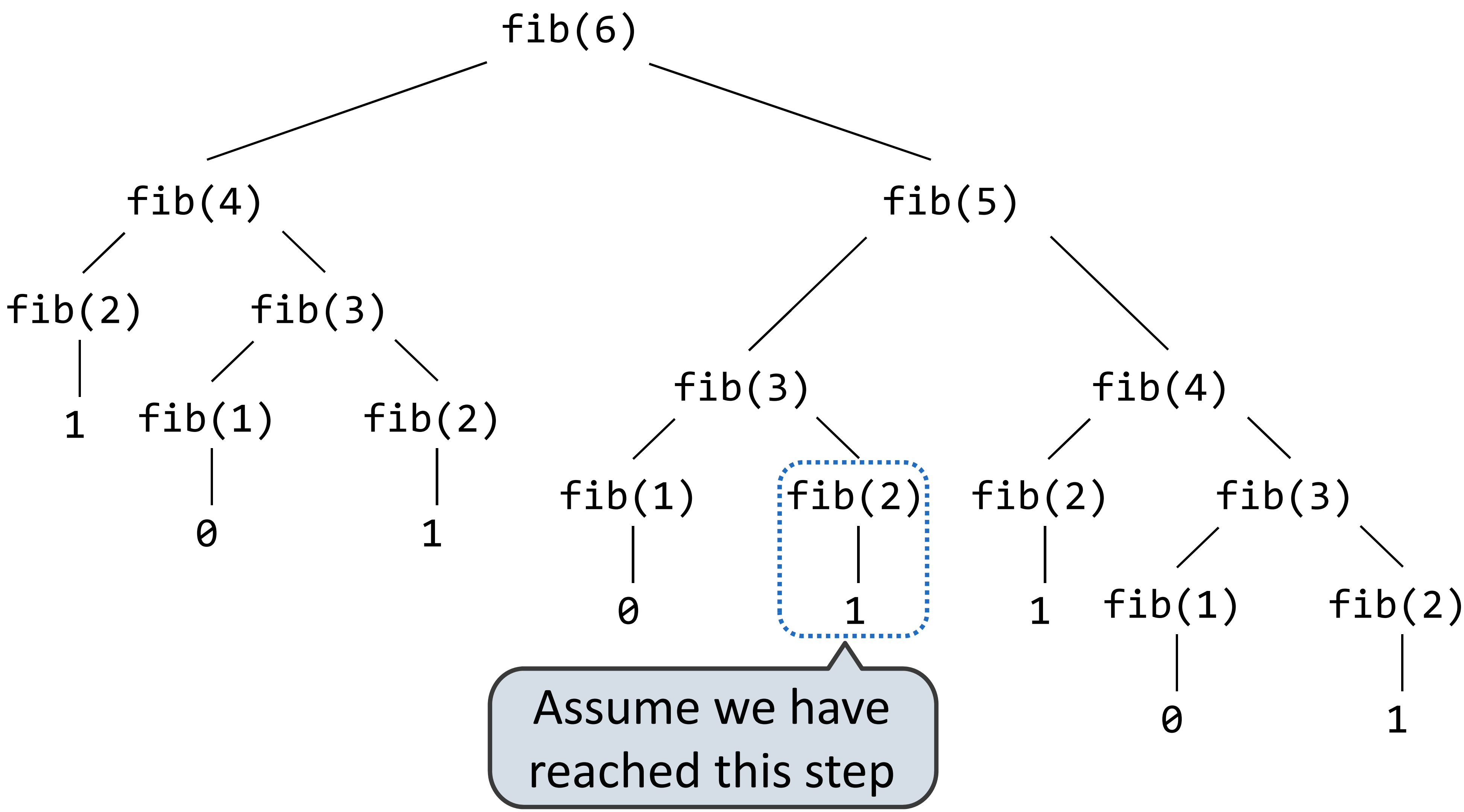

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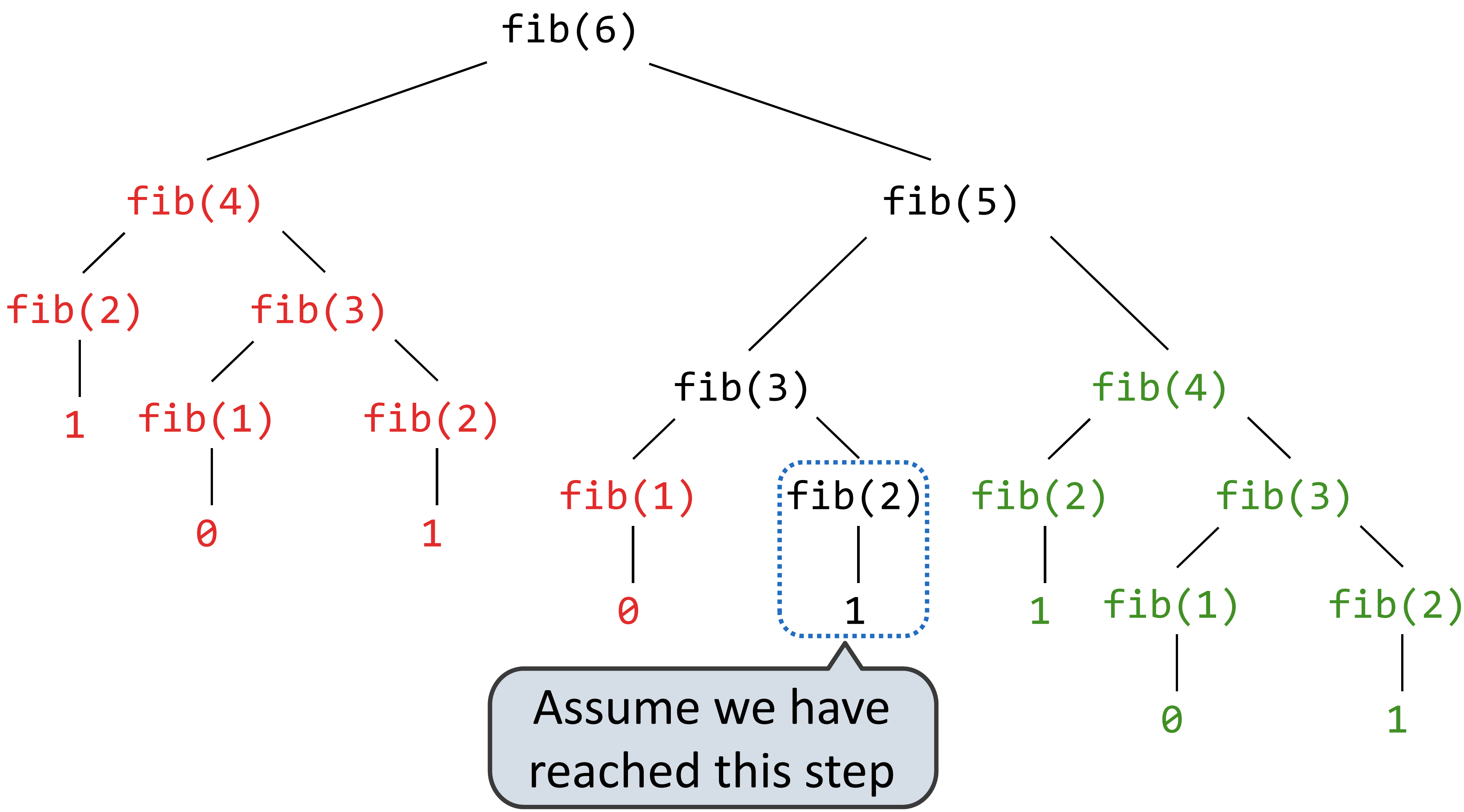

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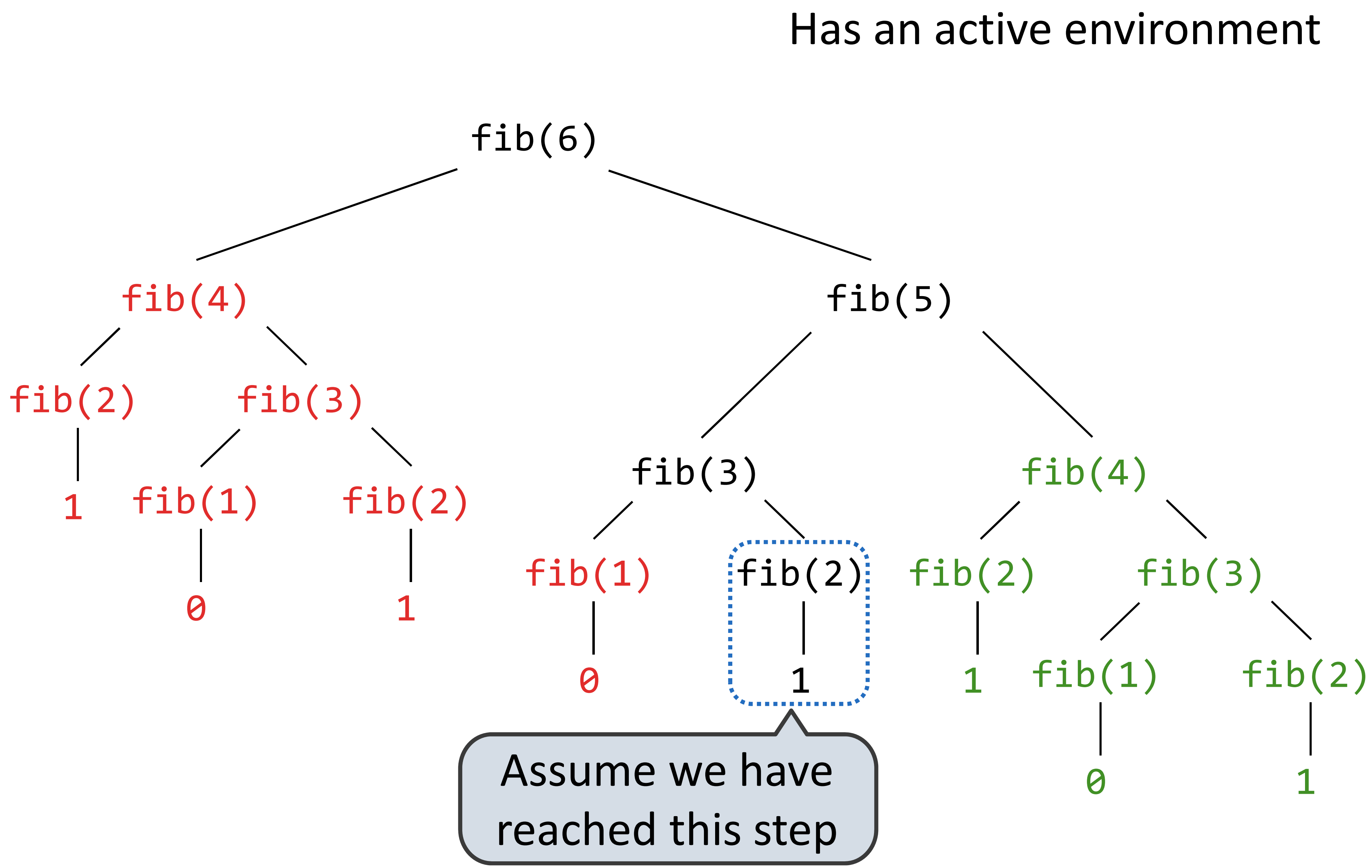

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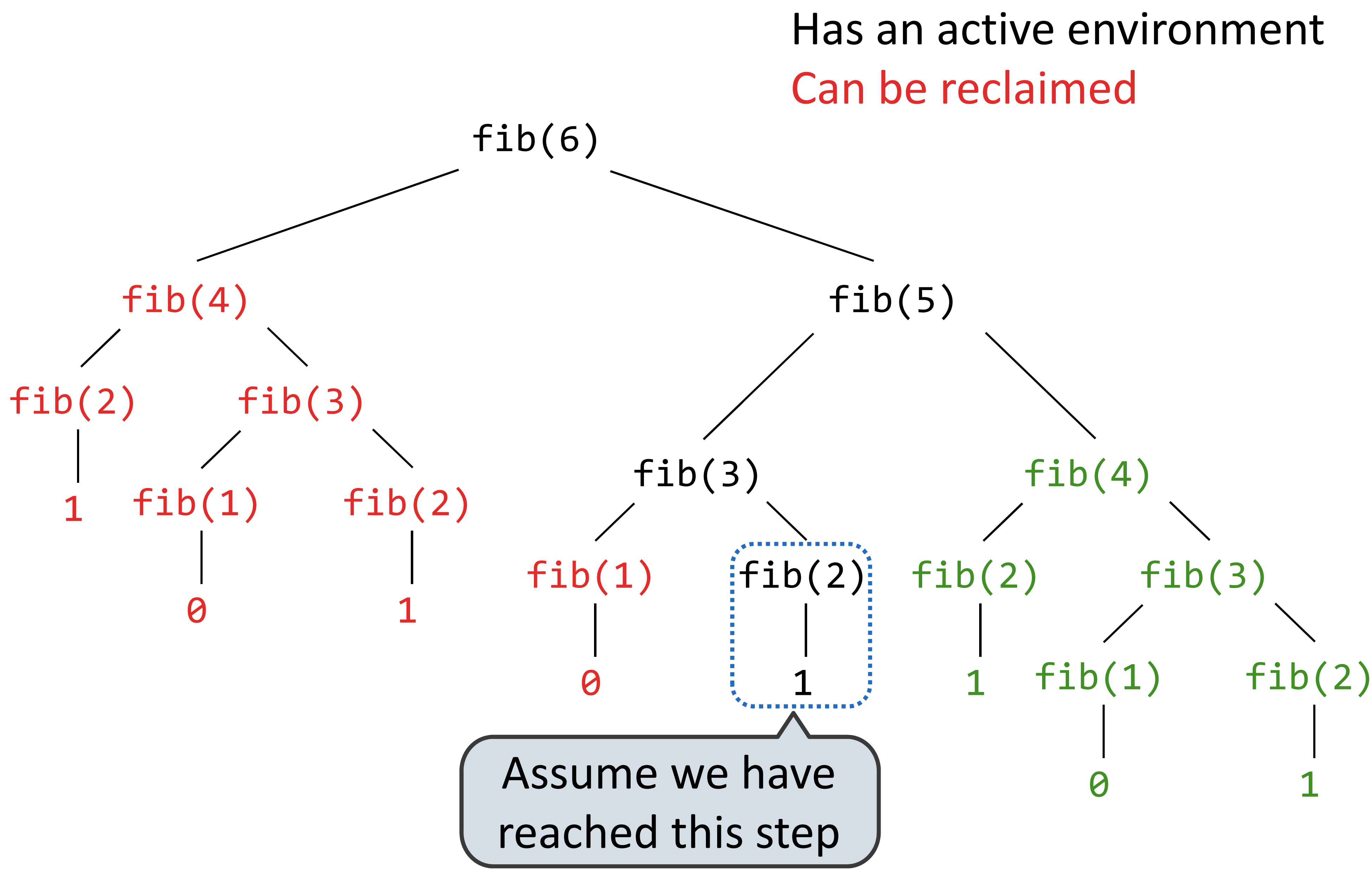
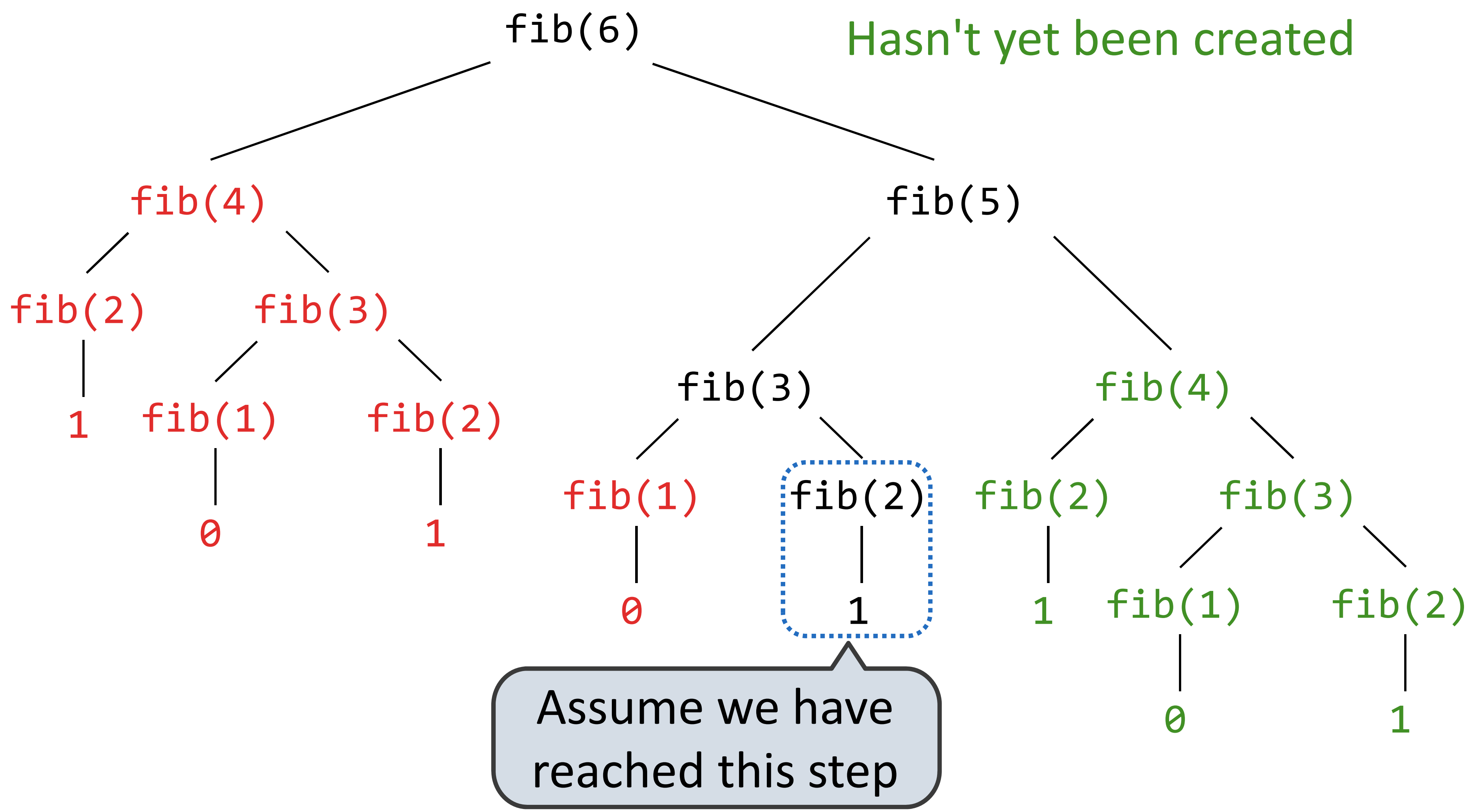


Fibonacci Memory Consumption

Has an active environment

Can be reclaimed

Hasn't yet been created



Iteration vs. Tree Recursion

Iterative and recursive implementations are not the same.

	Time	Space

```
def fib_iter(n):
    prev, curr = 1, 0
    for _ in range(n - 1):
         prev, curr = curr, prev + curr
    return curr


def fib(n):
    if n == 1:
        return 0
    if n == 2:
        return 1
    return fib(n - 2) + fib(n - 1)
```

Iteration vs. Tree Recursion

Iterative and recursive implementations are not the same.

	Time	Space
`fib_iter`	$\Theta(n)$	
`fib`		

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	Time	Space
fib_iter	$\Theta(n)$	$\Theta(1)$
fib	$\Theta(\phi^n)$	

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Next time, we will see how to make recursive version faster.

Comparing Orders of Growth (*n* is problem size)

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$\Theta(b^n)$

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$\Theta(b^n)$ Exponential growth! Recursive fib takes

$\Theta(\phi^n)$ steps, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61828$

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Incrementing the problem scales *R(n)* by a factor.

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Doubling the problem only increments *R(n)*.

$\Theta(1)$ Constant. The problem size doesn't matter.

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$\Theta(n^6)$ → Incrementing the problem scales *R(n)* by a factor.

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